FAST INTEGRAL EQUATION ALGORITHM FOR LARGE SCALE SIMULATION OF ACOUSTIC FIELD PROPAGATION IN BIOLOGICAL TISSUES*

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ABSTRACT
We describe elements of a volumetric integral-equation-based algorithm applicable to accurate large scale simulation of propagation of sound waves through inhomogeneous media. The primary goal of our work is to construct an efficient and high fidelity numerical simulation tool for investigating such effects as, e.g., acoustic energy transfer to the inner ear via non-airborne pathways. The considered algorithm makes possible simulations involving realistic geometries characterized by highly sub-wavelength details and large density contrasts, and described in terms of several million unknowns. Its two main elements are:
1. A fast and non-lossy Fast Fourier Transform based matrix compression method (AIM), previously developed for solving large scale electromagnetic problems and characterized by a competitive, $O(N \log N)$, solution complexity, where N is the number of unknowns.
2. A suitable solution scheme consisting of (i) casting integral equations into a form which exhibits separation of volume and surface components, (ii) solving the surface problem, (iii) solving the volumetric problem with the new source term modified by the surface component of the solution, and (iv) constructing the final solution by the proper recombination of surface and volume solutions. The implemented procedure is rigorous and does not suffer from ill-conditioning in the limit of large density contrasts.
Examples involving several million of unknowns and demonstrating acoustic field distribution in human head are presented.

INTRODUCTION
There is considerable interest and need for developing reliable and computationally efficient solution methods for elasto-acoustic problems characterized by high geometrical and material complexity. The applications include, e.g., the ability to understand and numerically simulate mechanisms of noise propagation through the human body, in particular through the human head, over a wide frequency range, with the objective to assess the noise induced damage to the human hearing system and, ultimately, to devise measures of the noise reduction.
It is a well established fact that integral-equation formulations provide the most accurate solutions to wave problems. They require, however, solving dense systems of linear equations. Traditional methods of solving such systems were characterized by the computational complexity and memory requirements of the order of N^3, where N is the number of unknowns corresponding to the object discretization of approximately ten points per wavelength. Hence, despite the reliability and accuracy of the integral-equation based methods, they would become computationally prohibitively intensive to provide solutions for realistic problems of interest.
During the last decade a significant progress has been made in the development of ***f***ast frequency- and time-domain integral-equation solvers and, as the result, the ability of accurate and fast numerical simulation of wave propagation and scattering in complex media has dramatically improved. New matrix compression techniques, the Fast Fourier Transform (FFT) base Adaptive Integral Method (AIM) [1] and the Fast Multipole Method (FMM) [2], have been

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developed, which allow solving large lsets of linear equations equations with dense matrix utilizing storage and execution times characteristic of problems involving sparse linear systems. The physical idea behind the compression methods is that interactions at large distances require less resolution than interactions at small distances. As the result, the computational complexity and memory requirements of the compression methods scale approximately linearly with the number of unknowns N.

Recently we initiated the development of an integral equation solver for the modeling of acoustic/elastic wave propagation through the human head and its neighboring tissues (the human torso). The solver is based on the volumetric integral equation formulation and it is applicable to simulation of acoustic/elastic wave propagation in general inhomogeneous media with complex geometry and material details. The solver exploits an additional important feature of the FFT-based AIM matrix compression method, which is its applicability to solve problems involving a broad range of wavelength, in particular the sub-wavelength, scales (i.e. involving geometry details significantly, say hundred times, smaller than the propagating waveform wavelength).

One of possible applications of the solver involves simulation of the energy transfer to the inner ear and assessment of relative importance and frequency dependence of such an energy transfer through multiple pathways associated with bone-conducted sound transmission.

The main subject of this paper is the description of the modifications of the volumetric integral equation formulation which will enable us to solve efficiently problems involving high contrast acoustic/elastic media. The method is based on a suitable rearrangement of the terms in the pertinent integral equation and it results in a two stage solution scheme which we describe below. We also provide examples illustrating the method effectiveness.

INTEGRAL EQUATIONS IN ACOUSTICS FOR MEDIA WITH VARIABLE COMPRESSIBILITY AND DENSITY

Our approach is based on the Lippmann-Schwinger (L-S) equations for acoustic waves propagating in an object characterized by spatially variable density $\rho(\mathbf{r})$ and compressibility $\kappa(\mathbf{r})$, immersed in a homogeneous "background" medium of constant density ρ_0 and compressibility κ_0.

The standard form of the L-S equation [3] can be obtained from the conventional differential equation of acoustics, rewritten in the form

$$\left(k^2+\nabla^2\right)p-k^2\left(1-\frac{\kappa}{\kappa_0}\right)p-\nabla\cdot\left[\left(1-\frac{\rho_0}{\rho}\right)\nabla p\right]=0, \tag{1}$$

where $k=\sqrt{\rho_0\kappa_0}\,\omega$ is the wave number in background medium.

Eq.(1) leads to the L-S equation

$$\begin{aligned} p^{(\mathrm{inc})}(\mathbf{r}) = p(\mathbf{r}) + \int_{\bar{V}} \mathrm{d}^3 r' \, \{k^2 g(\mathbf{r}-\mathbf{r}')\left(1-\frac{\kappa(\mathbf{r}')}{\kappa_0}\right)p(\mathbf{r}') \\ -\left[\nabla_{\mathbf{r}'} g(\mathbf{r}-\mathbf{r}')\right]\cdot\left(1-\frac{\rho_0}{\rho(\mathbf{r}')}\right)\nabla_{\mathbf{r}'} p(\mathbf{r}')\}, \end{aligned} \tag{2}$$

where $p^{(\mathrm{inc})}$ is the incident wave, satisfying the Helmholtz equation in the background medium,

$$\left(k^2+\nabla^2\right)p^{(\mathrm{inc})}(\mathbf{r})=0, \tag{3}$$

and $g(\mathbf{r})=\dfrac{\mathrm{e}^{\mathrm{i}k|\mathbf{r}|}}{4\pi\,|\mathbf{r}|}$ is the corresponding scalar Green function,

The integral in Eq.(2) is taken over the region V in which $\rho(\mathbf{r}) \neq \rho_0$ or $\kappa(\mathbf{r}) \neq \kappa_0$.
An alternative form of the L-S equation, better suited for our purposes, can be derived from the differential equation (1) recast in the form

$$\left(k^2 + \nabla^2\right)\left(\frac{\rho_0}{\rho} p\right) + k^2\left(\frac{\kappa}{\kappa_0} - \frac{\rho_0}{\rho}\right) p + \nabla \cdot \left[\left(\nabla \frac{\rho_0}{\rho}\right) p\right] = 0. \tag{4}$$

The resulting L-S equation is then

$$p^{(\mathrm{inc})}(\mathbf{r}) = \frac{\rho_0}{\rho(\mathbf{r})} p(\mathbf{r}) + \int_{\overline{V}} \mathrm{d}^3 r' \{k^2 g(\mathbf{r} - \mathbf{r}')(\frac{\kappa(\mathbf{r}')}{\kappa_0} - \frac{\rho_0}{\rho(\mathbf{r}')}) p(\mathbf{r}') - [\nabla_{\mathbf{r}'} g(\mathbf{r} - \mathbf{r}')] \cdot \left(\nabla_{\mathbf{r}'} \frac{\rho_0}{\rho(\mathbf{r}')}\right) p(\mathbf{r}')\}. \tag{5}$$

Eq.(6) involves the gradient of the density, which is particularly useful in problems involving large density discontinuities. A simple case[1] is a region $V \subset R^3$ in which $\rho(\mathbf{r})$ is continuously differentiable ($\rho \in C^1(V)$), but has a discontinuity on the boundary $S = \partial V$. The integral operator in Eq.(5) can be then separated into its surface and volume parts,

$$\begin{aligned} p^{(\mathrm{inc})}(\mathbf{r}) = {} & \frac{\rho_0}{\rho(\mathbf{r})} p(\mathbf{r}) + k^2 \int_{\overline{V}} \mathrm{d}^3 r' g(\mathbf{r} - \mathbf{r}')(\frac{\kappa(\mathbf{r}')}{\kappa_0} - \frac{\rho_0}{\rho(\mathbf{r}')}) p(\mathbf{r}') \\ & - \int_{\overline{V}-S} \mathrm{d}^3 r' [\nabla_{\mathbf{r}'} g(\mathbf{r} - \mathbf{r}')] \cdot \left(\nabla_{\mathbf{r}'} \frac{\rho_0}{\rho(\mathbf{r}')}\right) p(\mathbf{r}') \\ & - \int_S \mathrm{d}^2 r' \hat{\mathbf{n}}(\mathbf{r}') \cdot [\nabla_{\mathbf{r}'} g(\mathbf{r} - \mathbf{r}')] \Lambda(\mathbf{r}') p(\mathbf{r}'), \end{aligned} \tag{6}$$

where $\hat{\mathbf{n}}$ is the exterior unit normal to the surface S and the function $\Lambda(\mathbf{r})$, $\mathbf{r} \in S$, is related to the discontinuity of $\rho(\mathbf{r})$ across the boundary S

$$\Lambda(\mathbf{r}) = \lim_{\varepsilon \to 0+} \left(1 - \frac{\rho_0}{\rho(\mathbf{r} - \varepsilon \hat{\mathbf{n}})}\right). \tag{7}$$

In Eq.(6) the last term represents explicitly the contribution of the boundary S.

A TWO-STAGE SOLUTION METHOD FOR LARGE DENSITY CONTRAST PROBLEMS

A peculiar feature of the integral equation (6) is that, for $|\rho/\rho_0| \gg 1$ and moderate values of the refraction index $n = \sqrt{\rho\kappa/(\rho_0\kappa_0)}$, the surface term (resulting from the gradient of ρ_0/ρ) becomes the dominating term. It follows that the matrix equation, obtained by discretizing Eq.(5), becomes ill-conditioned: the surface component of the solution is well defined, but the volume part of the solution (pressure in the interior of the object) is poorly determined.
We can, however, remedy the difficulty and turn the situation to our advantage by separating the solution into its surface and volume parts, renormalizing them, and solving the problem in two stages, both without introducing any approximations (such as an expansion in ρ_0/ρ). We give below the outline of the procedure:[2]

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1 Our approach can be generalized in a straightforward way to problems with multiple discontinuity surfaces (interfaces).
2 Results such as the existence and uniqueness of the solutions can be proven rigorously under

We first observe that, in the limit $\rho/\rho_0 \to \infty$ $(\Lambda \to 1)$ the surface operator in Eq.(6) becomes the double-layer operator. Accordingly, we define a surface operator $K^{(0)}$ by

$$(K^{(0)}\, p)(\mathbf{r}) = -\int_S \mathrm{d}^2 r' \frac{\partial\, g(\mathbf{r}-\mathbf{r}')}{\partial\, n(\mathbf{r}')} p(\mathbf{r}') \qquad \text{for } \mathbf{r} \in V; \tag{8}$$

the operator $K^{(0)}$ maps functions p supported on S onto functions defined on V. In terms of this operator, we then define, on the boundary S, a function $p_{\mathrm{s}}^{(0)}(\mathbf{r})$ as the solution of the ***surface integral equation***

$$p^{(\mathrm{inc})}(\mathbf{r}) = (K^{(0)}\, p_{\mathrm{s}}^{(0)})(\mathbf{r}) \qquad \text{for } \mathbf{r} \in S_-, \tag{9}$$

where $S_- \subset V$ is a surface infinitesimally close to the boundary S and located inside the region V. We can then show that $p_{\mathrm{s}}^{(0)}$ is the solution of the exterior Neumann problem, $\partial p/\partial n = 0$ on S, and that Eq.(9) also holds ***everywhere in the region*** V, i.e.,

$$p^{(\mathrm{inc})}(\mathbf{r}) = (K^{(0)}\, p_{\mathrm{s}}^{(0)})(\mathbf{r}) \qquad \text{for } \mathbf{r} \in V. \tag{10}$$

Our next step is to represent the operator K as

$$K = K^{(0)} + \xi\, K^{(1)} \tag{11}$$

where the small parameter ξ is chosen so that the operator $K^{(1)}$ (***defined*** by Eq.(11)) remains finite in the limit $\rho/\rho_0 \to \infty$; we may, e.g., take ξ as the average value of $\rho_0/\rho(\mathbf{r})$ over the region V.

Having specified $K^{(0)}$ and $K^{(1)}$, and the solution $p_{\mathrm{s}}^{(0)}$, we represent the full solution as

$$p = p_{\mathrm{s}}^{(0)} + \xi\, p_{\mathrm{s}}^{(1)} + p_{\mathrm{v}}, \tag{12}$$

where $p_{\mathrm{s}}^{(1)}$ is the correction to the surface solution $p_{\mathrm{s}}^{(0)}$ and p_{v} is the volume part of the solution, defined such that

$$K^{(0)}\, p_{\mathrm{v}} = K_{\mathrm{s}}\, p_{\mathrm{v}} = 0 \tag{13}$$

(here K_{s} is the surface part of the integral operator of Eq.(6)).

By substituting Eqs. (11) and (12) into the original equation, $p^{(\mathrm{inc})} = K\, p$, and making use of Eqs. (10), (11), and (13), we find the ***v***olumetric integral equation

$$([K^{(0)} + \xi\, K^{(1)}]\, p_{\mathrm{s}}^{(1)})(\mathbf{r}) + (K^{(1)}\, p_{\mathrm{v}})(\mathbf{r}) = -(K^{(1)}\, p_{\mathrm{s}}^{(0)})(\mathbf{r}) \quad \text{for } \mathbf{r} \in \overline{V} \tag{14}$$

for the unknown fields $p_{\mathrm{s}}^{(1)}$ and p_{v}, with the r.h.s. expressed in terms of the previously determined surface problem solution $p_{\mathrm{s}}^{(0)}$. We stress that Eq.(10) (i.e., extension of Eq.(9) to the interior of the region V) is necessary for cancellation of large terms $\sim \xi^{-1}$, which would be otherwise present in Eq.(14).

The entire solution procedure consists thus of: (1.) solving the ***s***urface problem of Eq.(9), (2.) using the solution $p_{\mathrm{s}}^{(0)}$ to compute the r.h.s. of Eq.(14), (3.) solving the ***v***olumetric problem of Eq.(14) for $p_{\mathrm{s}}^{(1)}$ and p_{v}; and (4.) constructing the full solution according to Eq.(12).

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the usual regularity assumptions on the surface S and the functions ρ and κ.

We stress that neither of the equations (9) and (14) suffer from ill-conditioning in the limit $\rho/\rho_0 \to \infty$ ($\xi \to 0$). At the same time, since no small-ξ approximations were made, the procedure can be used for any material density.

STIFFNESS MATRIX COMPRESSION AND FAST SOLUTION METHODS

In view of the complexity of realistic anatomical models of the human head and the resulting problem sizes (up to millions of unknowns), compression of the stiffness matrix and a fast solution method are necessities. A technique particularly well suited to the considered type of problems (volumetric, subwavelength discretization) is the FFT-based AIM matrix compression [1]. We adapted this approach, originally developed for electromagnetics, and found it efficient in the case of acoustics. The current version of the code (partly parallelized for shared-memory systems) solves a typical problem of 1,000,000 unknowns (tetrahedra) in about 2 hours on 8 processors, and requires less that 7 GB total memory. We note here also that inhomogeneity of the scatterer does not add to the computational cost.

NUMERICAL EXAMPLE

As the first example of application of the described solution method we show results of code validation for the case of a layered sphere, immersed in air, and consisting of two concentric shells.

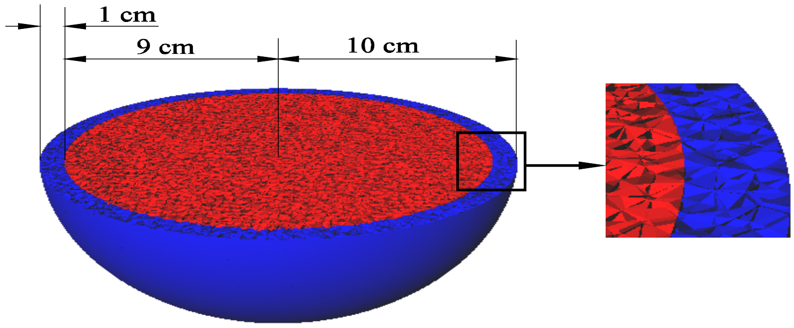

Figure 1. Discretization of the layered sphere with $N = 755,000$ tetrahedra.

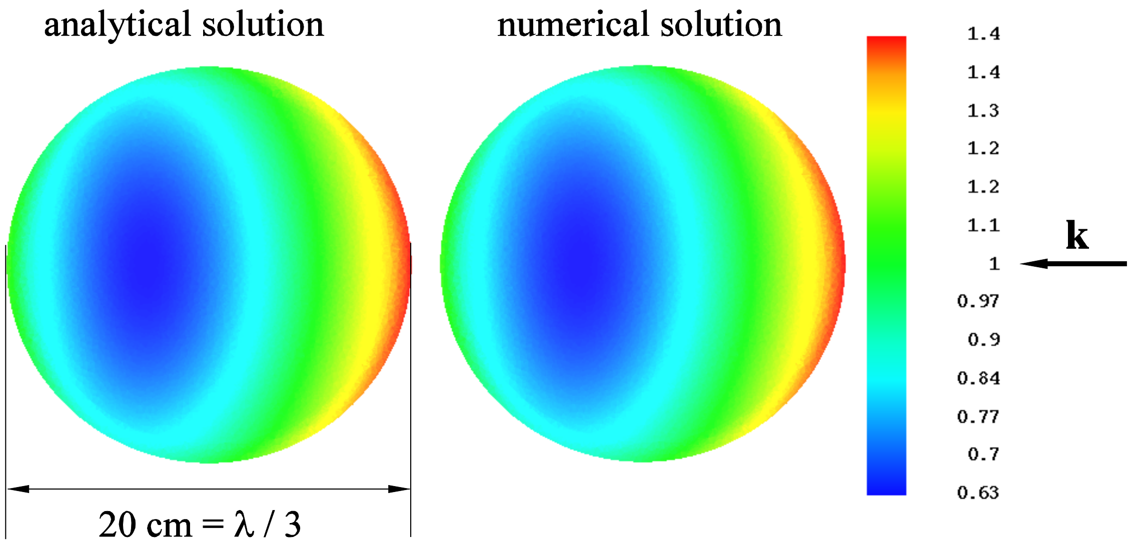

Figure 2. Distribution of the absolute value of the pressure, $|p(\mathbf{r})|$, on a plane parallel passing through the layered sphere center for a discretization with $N = 755,000$ tetrahedra.

The parameters of the outer shell are chosen to represent, approximately, the bone ($\rho/\rho_0 = 1777.2$, $\kappa/\kappa_0 = 6.5813 \cdot 10^{-5}$), and the interior of the sphere is described by mechanical parameters of water ($\rho/\rho_0 = 833.3$, $\kappa/\kappa_0 = 6.311 \cdot 10^{-5}$). The sphere geometry, shown in Fig. 1, consists of about 755,000 tetrahedra.

Fig. 2 shows distribution of the absolute value of the pressure, $|p(\mathbf{r})|$ for the incident wave with the wavelength $\lambda = 60\,\mathrm{cm}$, compared to the analytic solution. The errors in the computed pressure are of the order of 1 %.

As an example of a computation for a more realistic geometry, we show results for a human head model, represented by a tetrahedral mesh with about $N = 1,090,000$ tetrahedra. The material parameters are $\rho/\rho_0 = 10^3$, $\kappa/\kappa_0 = 10^{-4}$, $n \simeq 0.316$, and the wavelength is $\lambda = 60\,\mathrm{cm}$. Fig. 3 shows the distribution of the absolute value of the pressure, $|p(\mathbf{r})|$, in the axial plane, for the indicated direction of the incident wave. As before, the solution was obtained by means of the two-stage scheme. Convergence of the solutions was similar to that in the sphere problem.

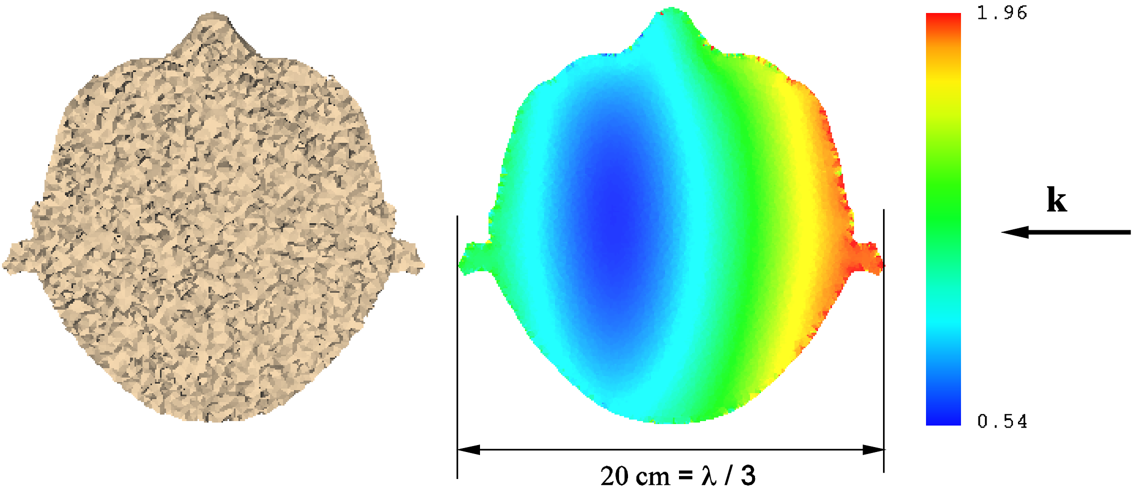

Figure 3. Distribution of the absolute value of the pressure, $|p(\mathbf{r})|$, in the axial plane, for a human head model. The computation was done for $N = 1,090,000$ unknowns with the two-stage solution scheme. A section of the tetrahedral mesh is shown to indicate the spatial resolution.

SUMMARY

We presented a novel approach to solving acoustic wave propagation in inhomogeneous media (such as biological tissues), based on re-formulation of the volumetric Lippman-Schwinger integral equation. The formulation, together with the fast FFT-based AIM solution method, result in an efficient numerical procedure whose cost is approximately linear in the number of unknowns and independent of the degree of inhomogeneity of the material. Results reported in the paper give us confidence that the described approach will provide valuable results in numerical simulations of sound propagation in the human head and in investigating its effects on the auditory system.

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