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ACOUSTIC BOUNDARY ELEMENT METHOD FORMULATION WITH TREATMENT OF NEARLY SINGULAR INTEGRANDS BY ELEMENT SUBDIVISION

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ABSTRACT
It is well known that the Boundary Element Method (BEM) in its standard version cannot readily handle situations where the calculation point is very close to a surface. These problems are found: i) when two boundary surfaces are very close together, such as in narrow gaps and thin bodies, and ii) when field points are calculated very close to the boundary. The difficulty is due to the near-singularity of the integrand, which causes failure of the numerical integration over the element. There are a number of techniques to overcome this problem, in many cases involving a reformulation of the integrand or the whole method. On the other hand, it is also possible to refine or improve the numerical integration, and maintain the standard BEM formulation. In this paper a numerical technique based on element subdivision, previously proposed by the authors, is made more general to cover most cases of interest. The subdivision is adapted to the strength of the near-singularity and is only performed when needed, not adding excessive calculation time and storage. The implementation is examined and verified with test cases.

INTRODUCTION
The Boundary Element Method is used in Acoustics for a wide range of problems where a sound field is created by scattering or radiation of sound.[1] It is particularly useful for infinite domains, due to its use of meshes over the domain boundaries rather than the whole domain, as it happens with the Finite Element Method (FEM). [2,3]

However, there are prices to pay for these advantages: BEM coefficient matrices are fully populated and therefore often require more storage space than FEM matrices, and more time consuming solving methods. Besides, the integrals in the standard BEM formulation have kernels that are singular when the collocation point is placed on the integrating element. This difficulty can be dealt with either with modified formulations or by means of singular numerical integration. Another situation arising for exterior domains is the instability caused by resonances of the corresponding interior domain, usually termed as the non-uniqueness problem.

In this paper we are concerned with yet another difficulty in BEM, which shows up when a collocation point or a field point is *close* to the element but not directly on it: the kernel becomes near-singular and standard numerical integration tends to fail.

If the close point is a collocation point, then this point belongs to an element which is very close to the element showing the near-singular behaviour. This situation is found in geometries with thin parts or narrow gaps. Moreover, even if the near-singularity is dealt with, one finds instability of the BEM solution, which can only be avoided by turning the thin parts into simple surfaces of zero thickness. This is not always possible for thin bodies, and hardly practical for narrow gaps. The problem has been treated in different ways in the literature.[4,5]

If the close point is a field point, where the sound pressure is being calculated from the BEM solution on the domain boundary, the near-singular behaviour only affects the result at that point, meaning that it is problematic to observe the sound field in the close neighbourhood of the boundary. Modifications to the BEM formulation are proposed by some authors to solve this aspect of the problem.[6]

In general, one can divide the approaches to the problem of near-singular integrals in BEM into two kinds: i) modifications of the method to reduce or "weaken" the near-singularity, and ii) improvements in the numerical integration without modifying the kernel. The second approach is often dismissed as resource intensive and unreliable, without further discussion. The authors of this paper and more recently R. Visser, have shown, however, that improved numerical integration can work.[7,8]

In this paper an extension to the strategy in ref. [7] is presented, extending it to a general three-dimensional BEM. It is shown that this numerical integration technique is affordable computationally and yields reliable results for the BEM.

THE BOUNDARY ELEMENT METHOD IN ACOUSTICS: THE NEAR-SINGULAR PROBLEM

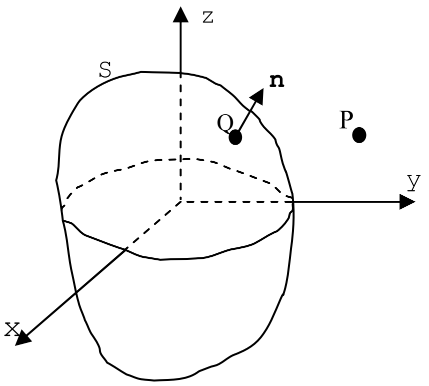

Figure 1. Generic integration domain and boundary surface.

The BEM approach to acoustic radiation and scattering problems is based on the Helmholtz Integral Equation that relates the pressure *p(Q)* and normal velocity *v(Q)* on the surface of a body of any shape (see figure 1) with the pressure at any point *p(P)* and the pressure of an incoming wave $p^I(P)$.[7] The harmonic time dependence $e^{i\omega t}$ is omitted, giving:

$$C(P)p(P) = \int_S \left(\frac{\partial G}{\partial n} p(Q) + ikz_0 v(Q) G \right) dS + 4\pi p^I(P) \qquad (1)$$

where *S* is the surface of the body, *Q* a point on that surface and *P* any exterior or interior point. The normal vector ***n*** is directed into the computational domain. The factor *C(P)* is the geometrical constant and represents the exterior solid angle at *P*.

The Greens function for 3-D free space is

$$G(R) = \frac{e^{-ikR}}{R}, \qquad R = |P - Q| \qquad (2)$$

In BEM, the surface S is discretized into elements, resulting into a matrix equation:

$$\mathbf{C}\,\mathbf{p} = \mathbf{A}\,\mathbf{p} + i\,k\,z_o\,\mathbf{B}\,\mathbf{v} + 4\,\pi\,\mathbf{p}^I \qquad (3)$$

If which **p** and **v** denote the pressure and normal velocities at the nodes and matrices **A** and **B** contain integrals of the kernel functions defined in (1) and (2).The left-hand side can be subtracted from the diagonal of the first term in the right-hand side of (3). Then the pressure on the nodes can be expressed as a function of the normal velocity and/or the incident pressure, for radiation and scattering problems respectively, by solving the system of equations. The sound pressure on any point of the domain can then be obtained from the surface values of pressure and normal velocity by integrating (1) again. This is called the "direct collocation" method.[2]

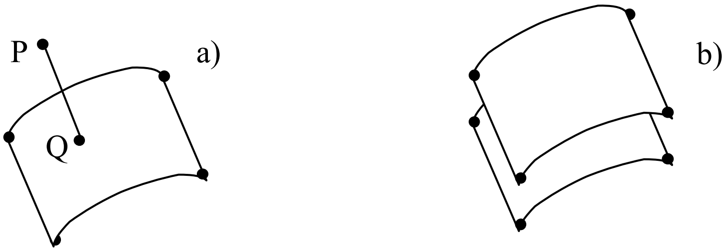

Figure 2. Representation of the near-singular problem: a) field point close to an element; b) close collocation points

By inspection of (1) and (2), it can be seen that the integrals corresponding to diagonal elements of A and B are singular when the collocation point is one of the nodes of the element to be integrated. However, they can be dealt with appropriate numerical integration rules.[9] On the other hand, sometimes the body has thin parts or narrow gaps, or domain points are very close to the boundary, as shown in figure 2. Then one finds that the distance R in (2) becomes very small, but not zero, and the integrands have a sharp peak. These are the near-singular integrals. The strength of the near-singularity depends on the relation between the distance point-element and the element size. In figure 3 this is represented for a generic line element. The integrand in BEM can be either a function of 1/R or $1/R^2$.

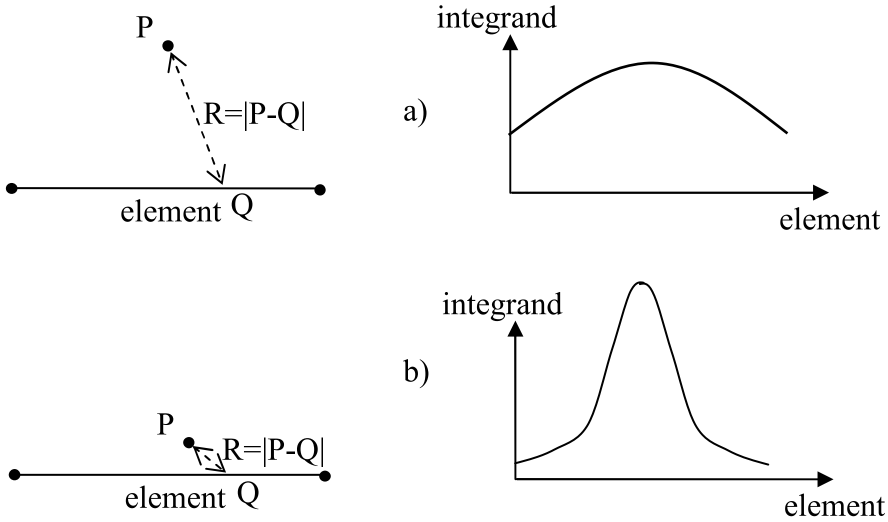

Figure 3. Generic representation of the increase of strength for a near-singular integrand, a) to b), as the near point gets closer to the element.

Numerical integration formulas such as Gauss-Legendre are commonly used in BEM. These rules yield exact results for polynomials up to a given order, depending on the number of integration points used. If the integrand is not a polynomial, the integration is approximately correct as long as its behaviour can be closely represented by polynomials of the allowed order.[9] In the case of near-singular kernels, there is a limit beyond which the integration fails, giving wrong results.

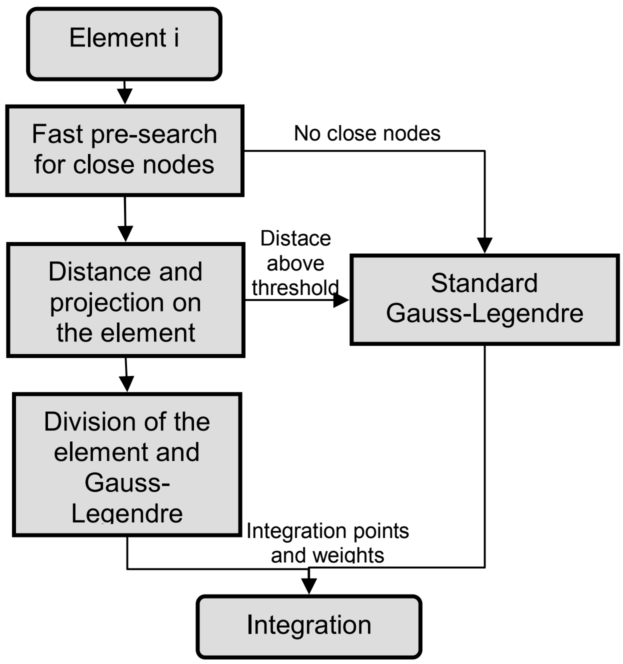

Figure 4. Flowchart of the method

TREATMENT OF NEARLY SINGULAR INTEGRANDS

The most obvious solutions to the near-singular problem sketched in the previous section would be: a) increasing the BEM mesh density wherever close points exist, making the elements smaller in size, and b) raising the order of the numerical integration. However, the first one increases the size of the BEM matrices and therefore computation time and storage significantly, while the second requires smooth high order derivatives.[9]

As an alternative to these trivial approaches, the use of adaptive integration algorithms has been proposed.[8] These algorithms divide the element into smaller integration intervals and evaluate the integral on them. The process is repeated recursively until the result converges below a given incremental threshold. The algorithm in [8] has been implemented for the purpose of comparison and it produces good results for the available test cases. It is almost as fast as the method developed for this paper. However, one can argue that adaptive algorithms depend on iterative evaluation of the integrand and assume no previous knowledge about the specific function to integrate, whereas the present method specializes to the strength of the near-singular integrals in BEM.

The method proposed here may be called, not adaptive, but *adapted* integration. The integrand is not evaluated repeatedly to achieve convergence. Instead the troublesome point on the element is found geometrically and element subdivision is applied to a depth that is proportional to the relative distance point-element. Integration points are assigned to each of the subintervals and then the integrand is evaluated on them only once.[7] A diagram of the method is in figure 4.

This method makes use of the available information about the nature of the integrand and the geometry of the setup, thus optimizing the resources. It does not add excessive overhead to the calculation, because the refinement only needs to be applied to the near-singular elements.

BEM integrals can be divided into frequency dependent and frequency independent parts, where the latter holds the near-singular kernels. In a frequency loop it is therefore only necessary to calculate near-singular integrals once.

ELEMENT SUBDIVISION IN TREE-DIMENSIONAL BEM

In this paper an extension to the ideas in [7] to general three-dimensional BEM is presented. The flowchart in figure 4 has been implemented into Matlab functions, and added to the existing OpenBEM formulation. The OpenBEM is a set of BEM formulations developed in Matlab by the authors and currently employed in a number of research projects.

First of all, the distance to the normal projection on the element from the close point is found, and also the local coordinates on the element of such projection, as depicted in figure 2-a. This can be done using geometrical rules for linear elements. For higher order elements, the distance is minimized using the element's shape functions. If the projection falls outside the element, the closest element point is found.

Distance and projection are passed to another function, the core of this method, which generates the integration points and weights for the near-singular integration. Both triangular and quadrilateral elements are implemented. Figure 5 shows a representation of integration points with and without near-singular integration.

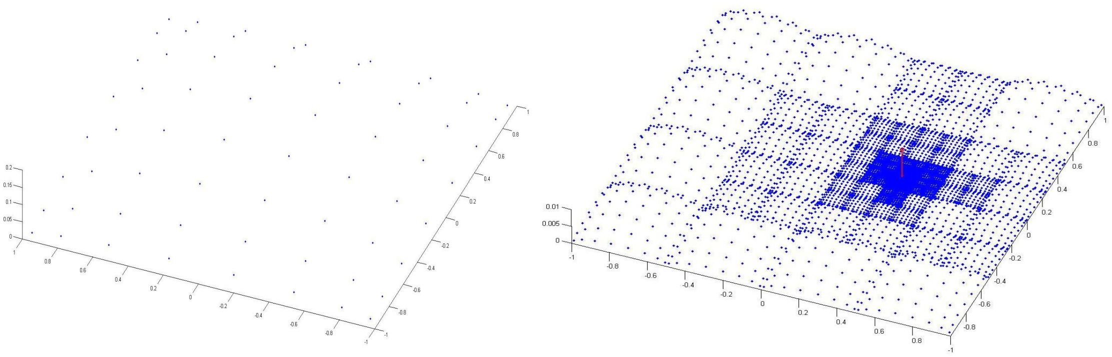

Figure 5. Gauss-Legendre integration points and weights (z value) on a generic surface quadrilateral element. a) Standard integration; b) Near-singular case, showing the near point.

The depth of the subdivision is a function of the distance from the close point to the element, relative to the element size. It has been decided that the smallest division must have an area smaller than the squared distance, based on the reasoning in the previous section (see figure 3). This choice gives good results for distances close to the machine's precision.

VERIFICATION WITH TEST CASES

In order to test the technique, a number of test cases with available analytical solutions have been tried: scattering of a plane wave by a sphere, a pulsating sphere, a first-order vibrating sphere, and a thin disc. Meshes with triangular and quadrilateral elements, linear or quadratic, have been used. We will present here only two cases: scattering of a plane wave by a sphere and by a disc, with isoparametric quadratic quadrilateral elements. Scattering solutions make use of the A matrix in equation (3), where the near-singularity is stronger. Besides, these examples cover both close field points and close collocation points.

Scattering sphere

This test is aimed to checking the ability of the method to cope with very close field points. The BEM equation is solved first for the surface nodes and then new coefficients are calculated, sometimes involving near-singular integrals, for the field points. Two variations have been tried: i) an arc of field points very close to the sphere's surface, covering θ values from 0 to π and a given value of φ in spherical coordinates (the plane wave arrives form the θ=0 direction) and ii) a line of field points approaching the sphere over a radial direction. The sphere mesh has 290 nodes and 96 elements.

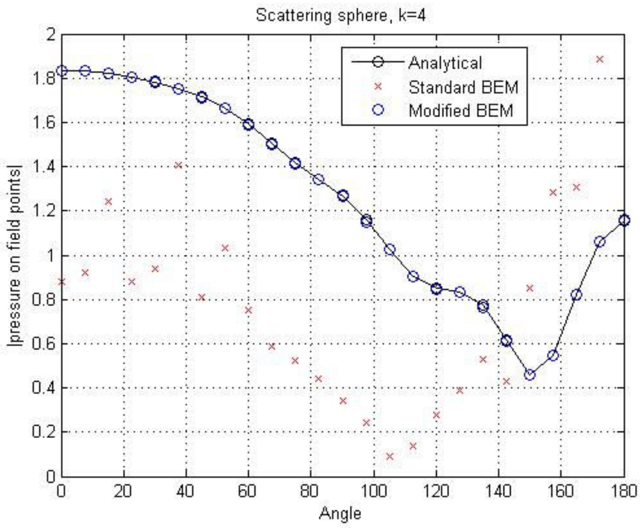

Figure 6. Modulus of the sound pressure on points on an arc close to the surface of a scattering sphere (k·radius=4). The normalized distance to the surface is 10^{-6}.

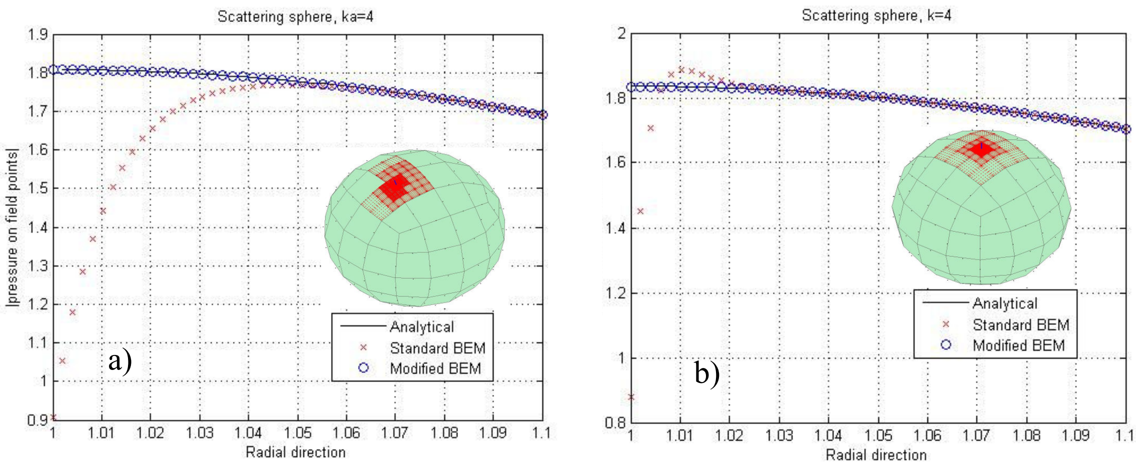

Figure 7. Modulus of the sound pressure close to a sphere of radius unity; a) approach towards some point inside an element, and b) approach to a node shared by four elements.

The first variation is represented in figure 6. Other calculations have shown that the method can cope with distances of almost the machine precision, around 10^{-14}. This is of course physically unrealistic, but it shows that all practical situations can easily be covered. Another variation is shown in figure 7: the points are placed on a line approaching the sphere surface. The breakdown is clear in the standard BEM, while the BEM with modified near-singular numerical integration can handle the close points.

Thin disc

The problem of close mesh surfaces, such as thin objects and narrow gaps, can also be dealt with using near-singular numerical integration.[7] The scattering of a plane wave by a thin disc has an analytical solution and can be used to test the method. The relative thickness of the disc is 10^{-6}, and the mesh has 560 nodes and 186 elements.

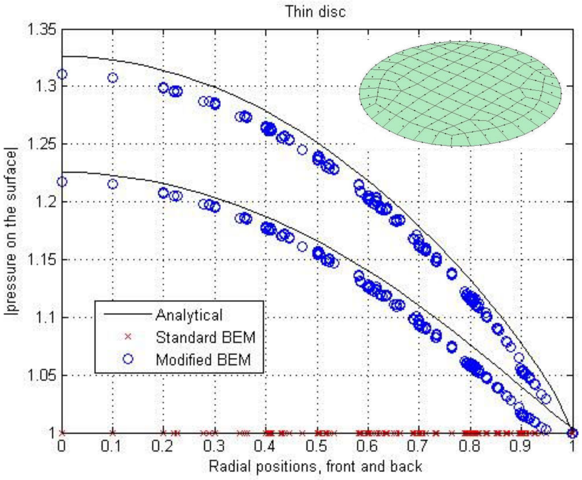

Figure 8. Modulus of the pressure on the surface of a thin disc.

The result is only approximately close to the analytical solution. The difference could be explained by number of reasons, other that the near-singular integration: i) insufficient mesh density, ii) difficulty to handle the sharp edge of the disc, iii) high condition numbers of the coefficient matrices (typical when meshes are close), and iv) inaccuracy of the analytical solution. All these possibilities have not been investigated thoroughly yet.

However, it is believed that the near-singular integration can be said to perform well, all in all.

CONCLUSIONS

An extension to a numerical integration technique for near-singular kernels, previously published by the authors, has been presented. These kernels are typical in the Boundary Element Method when field points or collocation points are very close to an element. The technique is verified using a number of test cases.

FUTURE WORK

A more complete convergence and error study will be presented, to complete the one included in [7]. More comparisons between adaptive integration and the present method will be run, to show the advantages and limitations of the two techniques.

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