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PHYSICAL MODELING STUDIES OF THE PIANO AND VIOLIN

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ABSTRACT
This paper reviews work by our group on the modeling of two different musical instruments using finite difference-time domain methods. The first project involves a fairly complete computational model of the piano, in which the motion of the hammers, strings, soundboard, and room air are all modeled, resulting in the sound pressure at the ears of a listener. The second project is a new study of bowed string motion, in which the motion of the string and bow are both modeled during Helmholtz motion.

I. INTRODUCTION
The modeling of musical instruments has been a topic of research for as long as scientists have studied musical acoustics, and many different approaches have been developed over the years. An increasingly popular approach is "physical modeling", which is based on equations of motion derived from Newton's laws. Work in this area has been reported for a number of different instruments, including the piano, guitar, violin, recorder, and various percussion instruments (unfortunately we do not have space here to give a complete references to this work). The relevant equations of motion have, of course, been known for a long time—the reason that physical modeling was not widely applied until recently is that it can only be implemented realistically by solving these equations numerically. It is thus the power of currently available computers that makes physical modeling possible—and since this computer power is rapidly increasing, it seems likely that physical modeling will be used more and more in the future.

While physical modeling of musical instruments is thus now feasible, we must still be clear on the goals of such work. What can be learned by physical modeling of a particular instrument? Since physical modeling can yield accurate results for the sound produced by an instrument, it can (if the computer and algorithm are fast enough) be used for real-time sound synthesis. Physical modeling thus has great potential in the area of computer music and computer-based musical instruments. However, this is not the primary goal of our own work in this field. Our aim is to use modeling to gain a better understanding of an instrument. Why does an instrument sound the way that it does? What aspects of the instrument are most important for the resulting tones? How would a hypothetical change in a particular component of the instrument change the tones that it produces or the way that it plays? These are some of the questions that can be addressed with physical modeling. In this paper we illustrate this "philosophy" with examples form our work on the piano and the violin.

II. FINITE DIFFERENCE-TIME DOMAIN MODELING
While all physical modeling is, in principle, based on Newton's laws, there are various ways to implement these laws in practice. Our own work employs a finite difference-time domain approach. The basic idea is to discretize both space and time, and use Newton's second law to derive a system of equations that express the state of the "system" (i.e., the instrument plus whatever else is being modeled) at the future times in terms of the system state at current and previous times. As a simple illustration, consider the motion of an ideal flexible string. In this case Newton's laws yield the usual wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad (1)$$

where y is the transverse displacement of a string that lies along the x direction, t is the time, and c is the wave speed. We now discretize space and time in "steps" of dx and dt, and write the displacement as $y(x,t) = y(\mathrm{i}dx,\mathrm{n}dt) = y(\mathrm{i,n})$, where i and n are integers. The derivatives in Eq. (1) can be written as finite differences with the result (see, for example, [1])

$$\frac{\partial^2 y}{\partial t^2} = \frac{y(\mathrm{i,n}+1) + y(\mathrm{i,n}-1) - 2y(\mathrm{i,n})}{dt^2} \qquad (2)$$

with a similar result for the other partial derivative in Eq. (1). Inserting these finite difference approximations for the derivatives into Eq. (1) we can then solve for the string displacement at the "next" time step n+1 in terms of the displacement at previous time steps. The result is

$$y(\mathrm{i,n}+1) = 2y(\mathrm{i,n}) - y(\mathrm{i,n}-1) + c^2 \frac{dt^2}{dx^2}\left[y(\mathrm{i}+1,\mathrm{n}) + y(\mathrm{i}-1,\mathrm{n}) - 2y(\mathrm{i,n})\right] \qquad (3)$$

Given the string displacement at time steps n and n–1, Eq. (3) can thus be used to find the displacement at step n+1, and then n+2, etc.

This is a very efficient and numerically stable way to treat an ideal string [1]. Of course, the equation of motion of a real string is more complicated than Eq. (1); the additional terms needed to account for string stiffness, damping, and other effects can all be included by expressing the relevant derivatives as finite differences, leading to equations similar to Eq. (3). There are different ways to express these derivatives; some approaches lead to an "explicit" set of equations, similar to Eq. (3), while others yield an implicit system of equations in which $y(\mathrm{i,n}+1)$ is expressed in terms of $y(\mathrm{i}\pm1,\mathrm{n}+1)$. Such implicit methods usually require the solution of a system of linear equations (a matrix inversion) for each time step, which is more complicated (numerically) than an explicit method such as Eq. (3). However, implicit methods often have significant advantages with regards to stability and accuracy. These issues have been discussed in the context of string simulations by Chaigne [2].

The equations of motion of other systems, including vibrating plates (soundboards), simple point masses (piano hammers), etc., can all be treated with a finite difference approach similar to that described above. In the next two sections we describe applications to the piano and the violin.

III. STUDIES OF THE PIANO

Over the course of several years, our group has developed a fairly complete finite difference-time domain model of the piano [3]. In this work we modeled the motion of the hammers, strings, soundboard, and room air. We did not include the motion of the piano action, but took as our initial condition a piano hammer moving with a specified velocity just prior to its collision with the strings. In our model the strings are excited by the collision with the hammer. The vibrating strings then exert forces on the bridge which sets the soundboard into motion. The movement of the soundboard then generates sound in the surrounding air.

The equations of motion and the associated finite difference equations (such as Eq. (3)) are all well known. However, these equations involve a large number of parameters, such as the Young's moduli of the strings and soundboard, the force-compression characteristics of the hammers, etc. These parameters were determined through experimental studies of the hammers, strings, and soundboard. These studies also served to test our models of these parts of the instrument. Our string modeling followed closely the work of Chaigne and Askenfelt [4], while the hammers were described by hysteretic force-compression characteristics proposed by Stulov [5]. The soundboard model was described in [6], and tested against measurements like those in [7].

The finite difference equations essentially treat a system as a collection of individual masses. In our work, each string was treated as a collection of ≈50-100 pieces, while the soundboard was about $100\times100=10^4$ individual vibrating elements. To achieve a useful frequency range, a room of typical size must be divided into 10^7 or more individual pieces. As a result, the room portion of the model is the most time consuming. Even so, this portion of the model is quite straightforward. For this part of the model we followed the work of Bottledooren [8] and treated the linear wave equation using finite differences. The equations of motion can be written in terms of the acoustic pressure p and the air velocity $\vec{v}$ as [9]

$$\rho \frac{\partial \vec{v}}{\partial t} = -\vec{\nabla} p \qquad (4)$$

$$\frac{\partial p}{\partial t} = -\rho c^2 \vec{\nabla} \cdot \vec{v} \qquad (5)$$

where c is the speed of sound. It is, of course, possible to eliminate the velocity to get a wave equation that involves just the pressure p. However, it is convenient to keep both of these variables, since the boundary conditions at the soundboard involve $\vec{v}$, and the boundary conditions at the walls of the room involve the acoustic impedance of the wall material [8,10], giving a relation between p and $\vec{v}$. Equations (4) and (5) can be readily solved using a finite difference approach on staggered grids [8,1]. An example of the results is given in Fig. 1, which shows the sound production efficiency p/v_b of a grand piano soundboard as a function of frequency. Here we have driven the soundboard by applying a sinusoidal force at the point where the strings for middle C contact the bridge; v_b is the velocity of the board at this driving point, and p is the amplitude of the sound pressure in the far-field, at a typical point in the virtual room. The ratio p/v_b is approximately constant over more than a decade of frequency, starting from just above 100 Hz. Hence, in this range it is a reasonably good approximation to take the sound pressure as proportional to the soundboard velocity. However, the sound pressure falls rapidly at lower frequencies. As is well known, this drop in the sound pressure occurs as the wavelength becomes larger than the size of the soundboard. Our calculation gives a quantitative description of this drop-off, and agrees well with experimental studies [11].

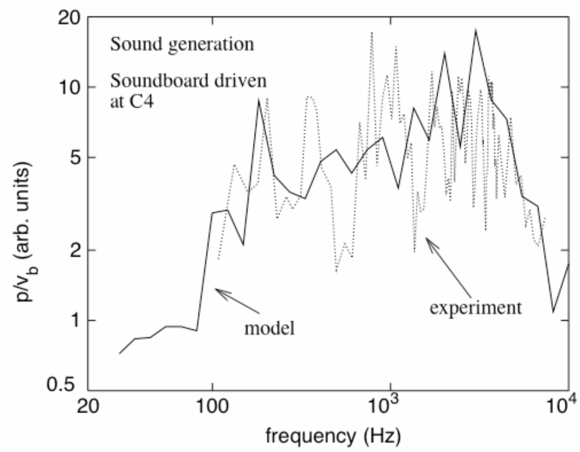

Figure 1. Sound production efficiency, expressed as the ratio p/v_b as a function of frequency for a grand piano soundboard. p is the amplitude of the sound pressure while v_b is the amplitude of the soundboard velocity at the driving point. The experimental results are from [11].

Results for piano tones from our model are given at www.physics.purdue.edu/piano/. Listening tests indicate that one of the most important parts of the model is the description of the hammers. Our current model uses the hysteretic function suggested by Stulov [5] with parameters determined by measurements in our lab on Renner hammers. This function provides a semi-quantitative, though far from perfect description of the measured force-compression characteristics [12,13]. While more work is needed to refine the modeling of piano hammers, these calculated tones are (in our view) very encouraging.

IV. MODELING OF THE VIOLIN STRING AND BOW

The motion of a bowed string has been studied extensively by many well-known scientists, including Helmholtz and Raman. It is an amazing fact that a bowed string, i.e., a string that is in constant contact with a bow of quite comparable mass, can vibrate at the precisely the same frequency as when the string is simply plucked. This was first explained by Helmholtz [14], who studied and explained the stick-slip motion of the string when driven by the frictional force from the bow. The periodic motion of bowed string of musical interest is commonly known as Helmholtz motion [15]. Most modeling studies of the bowed string start from D'Alembert's solution of the wave equation in terms of counterpropagating waves. Quantities such as the frictional force between the bow and string are then expressed as a sum of factors involving the bow and string velocities at previous times, corresponding to waves reflected from the ends of the string and returning to the bowing point (see, for example, [15,16]). For the simplest case, a perfectly flexible string with losses only at the ends, this gives a very simple and efficient way to model a bowed string. This approach can be extended to include string stiffness, torsional string motion, finite bow width, and other complications [16]. However, the simplicity and advantages of the D'Alembert approach are gradually lost as these complications are introduced. This has motivated us to investigate the use of finite difference-time domain methods for the bowed string.

Our modeling has been in conjunction with an experimental study of the friction force between the bow and string, using a glass rod as the "bow" [17]. One surprising result of this work (at least to us) is that the bow exhibits rather strong longitudinal oscillations. This raises the following question: what effect, if any, do these bow oscillations have on the establishment of Helmholtz motion in the string? Most previous studies of bowed strings have treated the bow as a simple "point" object moving at a constant velocity, or perhaps with a specified acceleration. While a few previous studies of bowed strings have included the bow in some way [16], we are not aware of any work that treats the bow in terms of its own realistic equation of motion. With this motivation, we have implemented a finite difference model for the combined motion of the string and the bow. In our initial work we have for simplicity assumed a perfectly flexible, lossless string (described by Eq. (3)) with a tension, length, and mass per unit length corresponding the E string of a violin, with loss only at the ends. We have also ignored torsional string motion. The bow is modeled as simple rod moving longitudinally using the known Young's modulus and density of glass, with a length equal to that of the rod used in our experiments (L = 0.90 m). The kinetic friction force was taken as an exponential function of the relative velocity between the bow and string.

Figure 2 shows results for the string velocity at the bowing point as a function of time. Here the bow-string contact point moved at a constant velocity—we refer to this as "no model of the bow". The results are typical of calculated bowed string motion using D'Alembert-type methods. For this choice of parameters Helmholtz motion was established at about t = 0.080 s; the last non-Helmholtz slip event is indicated by the arrows in Fig. 2. In these plots v_{string} = +0.20 m/s corresponds to sticking to the bow, while the negative values of v_{string} correspond to slip. The "noise" during slip is due to vibrations generated by the abrupt switching between sticking and slipping; these vibrations are eventually damped by loses as the bridge and nut, but are still significant at the early times shown here.

Figure 3 shows the behavior when the motion of the bow is included. The bow was given an initial velocity equal to that used in Fig. 2, and a constant force was applied so that the bow moved at a constant average velocity. The string again reaches a "good" Helmholtz motion, but now it reaches this state at about t = 0.063 s, which is roughly 20% sooner than in Fig. 2. This shows that the bow can have a significant effect on the establishment of Helmholtz motion. In the simulation of Fig. 3 the initial bow-string contact point was 0.90L from the frog end of the bow. Moving the contact point to 0.30L gave the results in Fig. 4. Now Helmholtz motion is established at about t = 0.069 s, which is somewhat longer than in Fig. 3.

These results are still preliminary, but they do seem to demonstrate that motion of the bow can have a significant effect on the "playablity" of a violin and bow. Further work is certainly needed—it will be interesting to see how the effects of string stiffness, torsional motion of the string, and more realistic models of the bow affect these results. We will, of course (!), incorporate these effects using a finite difference approach. While the "traditional" D'Alembert methods could probably be implemented, we believe that the finite difference method has particular advantages when including the dynamics of the bow.

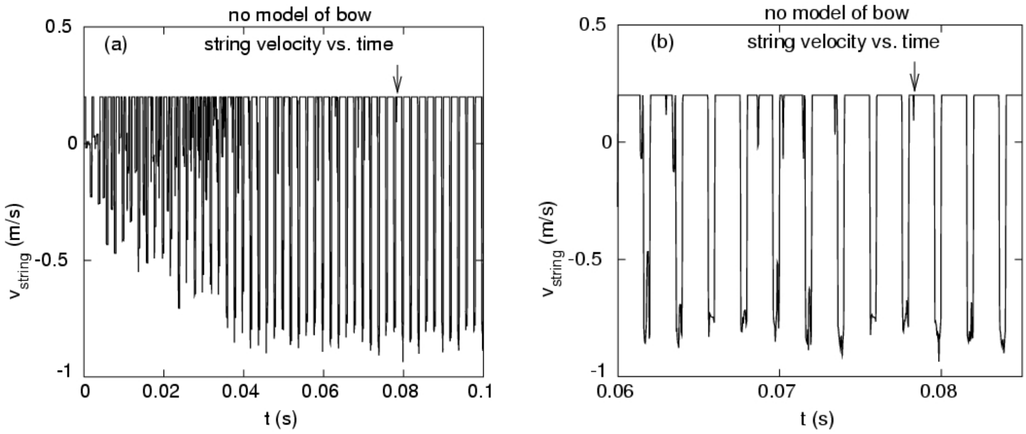

Figure 2. Motion of a bowed string. Here the bow-string contact point moved at a constant velocity. The arrows indicate the last non-Helmholtz slip event. The string exhibits pure Helmholtz motion at longer times.

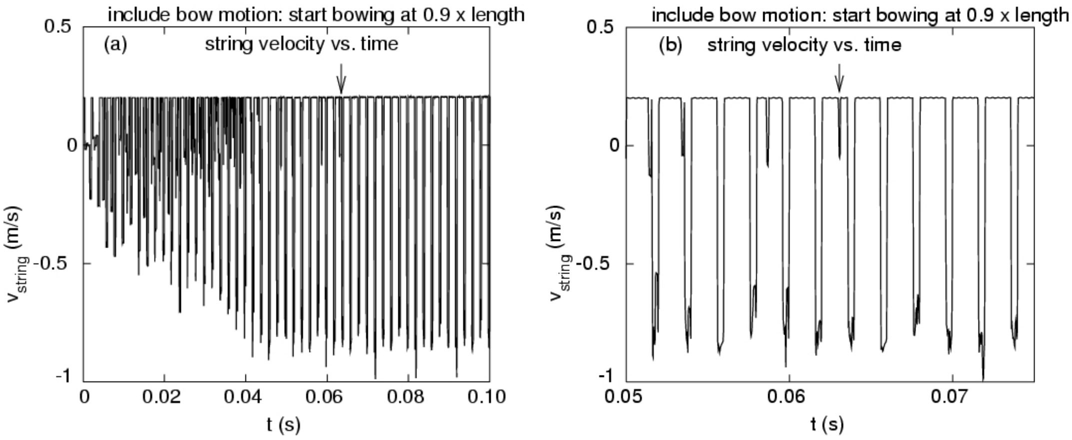

Figure 3. Motion of a bowed string with the motion of the bow included. The bow-string contact point was initially 0.90*L* from the frog end of the bow. The arrows indicate the last non-Helmholtz slip event. The string exhibits pure Helmholtz motion at longer times.

V. CONCLUSIONS AND OUTLOOK

We have given a brief review of the use of finite difference-time domain modeling as applied to the piano and the violin. We have shown that the sound produced by a musical instrument can be efficiently calculated by a finite difference treatment of the pressure and velocity in the air surrounding the instrument. We have also described how the finite difference method can be applied to study how the vibrations of a bow affect the motion of a bowed string.

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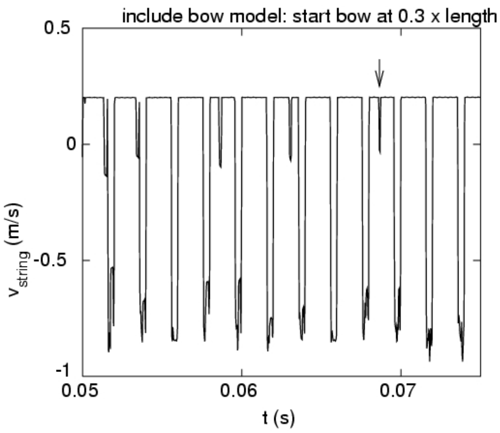

Figure 4. Same as Fig. 3; here again the motion of the bow is included. The bow-string contact point was initially 0.30*L* from the frog end of the bow. The arrow indicates the last non-Helmholtz slip event. The string exhibits pure Helmholtz motion at longer times.

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