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Lüke and Power Residue Sequence Diffusers

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ABSTRACT

A disadvantage of Schroeder diffusers is the "flat plate" effect that occurs when all the wells radiate in phase. For diffusers constructed using integer-based pseudorandom sequences to generate the well depths, this phenomenon takes place at a frequency that is the product of the diffuser's base design frequency and the prime number used to generate the sequence. Quadratic residue and primitive root diffusers' postponing this effect for higher frequencies requires longer sequences as the prime number of a sequence is restricted by its length, and this is often impractical. This problem can be reduced by using a sequence of small length that is based on larger prime number generator. The concepts of Lüke and power residue sequences as solutions are introduced and their characteristics are investigated. Their performances when applied to Schroeder diffusers are explored and comparisons with standard quadratic residue and primitive root diffuser are made. The results show the flat plate effect is moved to high frequencies but for Lüke sequences at lower frequencies, redirection rather than dispersion is achieved. Modulation is used to mitigate these problems.

INTRODUCTION

In the 1970s, Schroeder introduced the concept of using maximum length sequences in diffuser design to improve sound diffusion in concert halls and reverberation chambers[1]. Since then, a variety of diffusers has been developed[2]. A well known and widely applied class of diffusers is one that consists of different well depths which are generated using integer-based pseudorandom sequences such as Quadratic Residue (QRD) or Primitive Root Diffusers (PRD)[3].

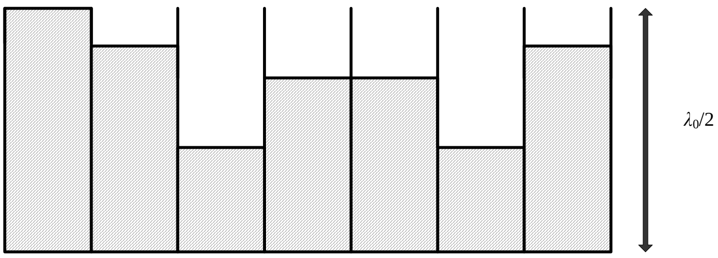

Figure 1. Schroeder diffuser generated using a Quadratic Residue Sequence of p=7.

A disadvantage that these diffusers display is that at specific frequencies their wells will radiate in phase and the whole structure will reflect sound as though it is a flat plate. The flat plate effect arises because there is a simple integer relationship between the different well depths. To illustrate this, consider the N=7 QRD which has a well depth sequence of s_{qrd}= [0,1,4,2,2,4,1] which are then converted into physical depths in the diffuser (Fig. 1). There will be a frequency at which the wells of depth 1 will be half a wavelength deep. At this frequency, all the reflected waves radiated from the diffuser will be in phase. The frequencies that this flat plate effect is noticed is usually pf_0, where f_0 is the diffuser's base design frequency and p is the prime number used to generate the pseudorandom sequence[2]. The prime number is directly linked with the period of the diffuser N for both QRD (N=p) and PRD (N=p-1). A solution to this problem is to move the first flat plate frequency to a higher frequency were it is of no concern. However, in order for QRDs and PRDs to do so longer sequences are required, which results in wider diffusers, or narrower wells. The experience of the last thirty years shows that narrow period diffusers are used much more often than wide diffusers, not least because narrower diffusers are cheaper to make and install.

A number of methods have been introduced in an attempt to solve this problem such as the use of non-integer based sequences[4], orthogonal modulation schemes[5] and numerical optimization[6]. In this paper, a way of mitigating this problem is presented by introducing integer-based sequences that are of small periods but generated using larger integers. Such sequences are Type-II Lüke and Power Residue sequences. The paper begins by outlining the Fourier prediction theory, which also enables the underlying principles to be understood. The general principles behind the sequences are then given. Then the performance of the sequences is considered using the Fourier Method. Finally modulation schemes are presented to mitigate the periodicity problems.

THEORY

Consider a structure with a distribution of reflection coefficients R_n throughout its surface. The surface is considered to be extruded in one direction, so that significant diffraction only occurs in one plane; this simplifies the prediction and interpretation of the results. Consequently, 1D or single plane diffusers are being considered. For normal incidence sound, the far-field reflected pressure, p, can be approximately found using[2]:

$$p(\theta) = \sum_{n=0}^{N-1} R_n e^{-inkw\sin(\theta)} \qquad \text{(Eq. 1)}$$

where θ is the angle of reflection, n is the well number, R_n is the reflection coefficient of the n^{th} well, k the wavenumber, w the well width and N the number of wells. Note that this is a Discrete Fourier Transform of the value $\Omega = kw\sin(\theta)$, and for this reason the prediction model is often referred to as a Fourier model.

SEQUENCES

Since the goal of a diffuser design is usually a uniform scattered pressure distribution, one structure that should diffuse well is one, which has reflection coefficients that display a uniform magnitude when Fourier Transformed. The Wiener-Khinchin theorem states that the square of the magnitude of the sequence's Fourier Transform is equal to the Fourier transform of its autocovariance (or autocorrelation) function. As a result of that, a sequence that generates a distribution of reflection coefficients whose autocorrelation function is a Kronecker delta function, will display good diffusion properties. Consequently, pseudo-random number sequences should be a good choice. The reflection coefficient is given for every sequence by the equation:

$$R_n = \exp\left[i2\pi \frac{s_n}{P}\right] \qquad \text{(Eq. 2)}$$

where s_n is the pseudo-random sequence and P is the integer the sequence is based on.

A. Type-II Lüke Sequences

Type-II Lüke sequences are generated in families of p-1 for any prime p by the equation[7]:

$$s_n^{(r)} = \left(\alpha^n (p-1) + rnp\right) \bmod p(p-1), \quad 0 \le n, r \le p-2 \qquad \text{(Eq. 3)}$$

where α is the primitive root of p, r denotes the family number of the sequence and n the coefficient number of the sequence. They are based on the integer $P=p(p-1)$ and have a period of $N=p-1$. The reflection coefficients of Type-II Lüke sequences have the following autocorrelation properties[7] :

$$\left|R_{r,r}(\tau)\right| = \begin{cases} p-1, & \tau = 0 \\ 1, & \tau \ne 0 \end{cases} \qquad \text{(Eq. 4)}$$

where τ is the autocorrelation delay variable. This autocorrelation function is the same with that of a Primitive Root sequence of the same period.

Essentially the sequences are formed by superposing the Primitive Root sequences, s_{prd} of prime p (Eq.5) and a steady step sequences s_{step} of the same period (Eq. 6):

$$s_{prd} = \alpha^n \bmod p \qquad \text{(Eq. 5)}$$

$$s_{step} = rn \bmod p - 1 \qquad \text{(Eq. 6)}$$

with r giving the step size. This is possible because a linear ramp can be added to any number sequence, provided the period is correct, without changing the autocorrelation properties[7]. So, Primitive Root sequences can be considered to be the first sequence (r=0) of each Type-II Lüke sequence family. From any Primitive Root sequence a set of p-2 new Type-II Lüke sequences can be generated, each one with a different step size.

To give an example for p=7, α=3 and s_{prd}= [3,2,6,4,5,1]. The arguments of the reflection coefficients of this sequence, at the design frequency f_0, are 2π3/7, 2π2/7, 2π6/7, etc. The flat plate effect will occur when all reflection coefficient are 1 (all their arguments are 0). This can be accomplished by multiplying them by a factor of 7 which happens when the incident wave is of frequency $f=7f_0$. On the other hand for p=7, α=3 and r=1 a Type-II Lüke sequence is generated s_{lsd}= [6,25,26,15,10,23]. The arguments of the reflection coefficients in this case, at the design frequency f_0, are 2π6/42, 2π25/42, 2π26/42 etc. For all the reflection coefficients in this case to become 1 their arguments need to be multiplied by a factor 42 which happens when the incident wave is of frequency $f=42f_0$. For a base design frequency of 500Hz, this corresponds to a flat-plate frequency of 21kHz.

B. Power Residue Sequence

For a prime number p that can be expressed in the form $p=MN+1$, where M and N are integers, M Power Residue sequences of period N can be generated using the equation[7]:

$$s_{pwr}^{(r)}(p,N) = (\alpha^r \beta^n) \bmod p\,, \qquad 0 \le r < M \ \text{ and } \ 0 \le n < N \qquad \text{(Eq. 7)}$$

where α is the primitive root of p and β is α to the power of M ($\beta=\alpha^M$).

A set o N integers D= [d_1,d_2,…,d_N] that are modulo an integer p are said to form an integer difference set (p,N,λ) if every integer h≠0 can be expressed in exactly λ ways by:

$$d_i - d_j \equiv h \bmod p \qquad \text{(Eq. 8)}$$

If the Power Residue sequence forms a cyclic (p,N,λ) difference set then the reflection coefficients that it generates display autocorrelation values[7]:

$$|R_{r,r}(\tau)| = \begin{cases} N, & \tau = 0 \\ \sqrt{N - \dfrac{N}{M} + \dfrac{1}{M}}, & \tau \ne 0 \end{cases} \qquad \text{(Eq. 9)}$$

where τ is the autocorrelation delay variable. It is important to note that these sequences display worse autocorrelation properties from Quadratic Residue, Primitive Root and Type-II Lüke sequences as the out of phase value is always greater than 1 and becomes grater as p increases.

Essentially Power Residue sequences are under-sampled Primitive Root sequences, with a sample taken every M^{th} coefficient, using a different starting point, which is set by r. For example, for p=11, α=2, and s_{prd}= [1,2,4,8,5,10,9,7,3,6]. If M=2, two Power Residue sequences are constructed, starting from the first coefficient (r=0) to form the Power Residue sequence s_{pwrd}= [1,4,5,9,3] and from the second (r=1) to form s_{pwrd}= [2,8,10,7,6].

There are three cases that form cyclic difference sets and need to be considered[7]:

1. M=2 and N is odd
2. M=4 and $N=n^2$ where n is odd
3. M=8 and $p=8n^2+1=64m^2+9$ where n and m are odd

Since the goal is to postpone the flat plate effect for higher frequencies the most promising cases is the last as it accomplices higher primes p with the shortest sequences possible. The case that falls under this category that generates a suitable length period is p=73 (M=8, N=9). The higher prime number generator means the first flat plate frequency is at 73 times the design. Such a diffuser would be less than 40*cm* long per period. In order of accomplishing that using QRD or PRD one would need periods N_{qrd}=73 and N_{prd}=72 which would result in a single period of the diffuser to be almost 3 meters long.

DIFFUSER EVALUATION

To evaluate the performances of the new sequences, diffusers were simulated and predictions of their scattered pressure distribution made. Their diffusion coefficient[2] was calculated in order to both aid the interpretation of the results and to form comparisons with other integer based diffusers such as the standard QRD and PRD.

A. Type-II Lüke Diffusers

The diffusers that are generated with steady step sequences of opposing inclinations can be paired as they perform similarly. This singles out one sequence that cannot be paired, the middle one which is generated for $r=M/2$.

The case of Lüke Sequence diffusers (LSD) generated by the integer P=42 is considered. They are diffusers of period N=6 and well width approximately 4.2*cm*. For the second LSD of the family (r=1) the maximum depth is 42.1*cm* for a design frequency of f_0=500*Hz*. In order of having a better understanding of how such diffusers will perform when applied in a room 8 periods of diffuser are considered (overall size 2*m*).

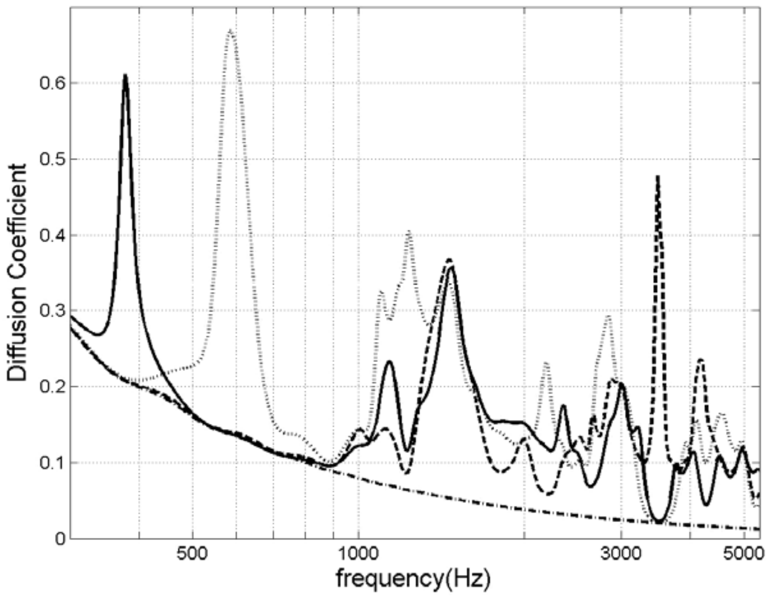

Figure 2: Prediction of the autocorrelation diffusion coefficient of 8 period diffusers' (p=7) with design frequency 500*Hz* using the Fourier Method of LSD (r=1) ——, LSD (r=3) — —, PRD ·······, Plane Surface -·-·-·.

The diffusion coefficient of the diffusers will be used initially to estimate the overall performance of the diffusers. The diffusion coefficient takes up values from 0 to 1, with 0 stating perfect reflection while 1 stating uniform scattered distribution in all angles of reflection. Figure 2 displays the diffusion coefficients of some diffusers of this family of LSD along with the equivalent PRD of the same characteristics. The PRD is expected to display a flat plate effect at $7f_0$=3.5*kHz* while the LSD diffusers are expected to display their first flat plate effect at $42f_0$=21*kHz*. As is evident from this figure the LSD s_{lsd} (r=1) seems not to be meeting the demand of postponing the flat plate effect as it displays a dip in the diffusion coefficient similar to the PRD's flat plate effect at 3.5*kHz*.

Figure 3 shows the scattered distribution from 2 periods of LSD of (r=1) in comparison to the scattered distribution from a plane surface at the problematic frequency of 3.5*kHz*. At this frequency, the LSD appears to be redirecting instead of scattering the incident wave. The reflection coefficients at 3.5*kHz* display arguments 0, $\pi/3$, $2\pi/3$, π etc. They display a phase change of $\pi/3$ from one to the next. The gradual offset of the reflection coefficients mimics the gradual phase offset used to beam steer loudspeaker arrays, and this is why the main reflected lobe is redirected into another direction. A sequence of such reflection coefficients can be derived from sampling a flat plate that is at an angle of the incident wave which leads to the conclusion that LSD at 3.5*kHz* act as such a titled flat plate.

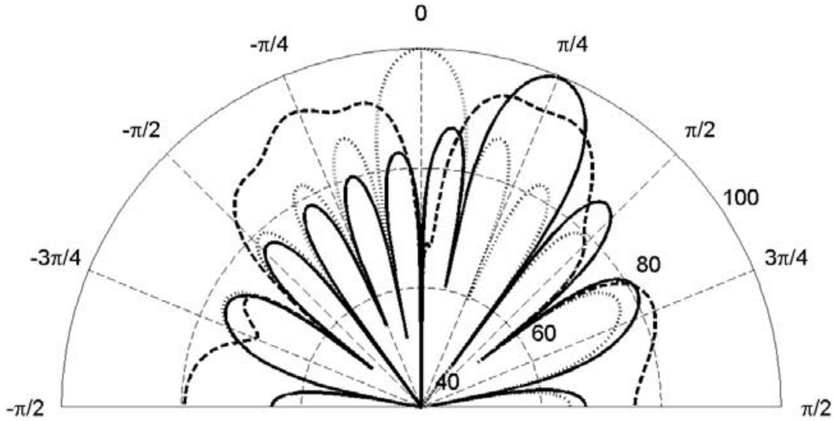

Figure 3: Scattered intensity distribution (*dB*) at the tilted flat plate frequency for 2 periods of LSD (*N*=7,*r*=1) periodic - - - - - - and modulated with its inverse ——— in comparison with a plane surface of equal length · · · · · · · ·.

The only case within each LSD family that seems to escape the tilted flat plate effect is the middle one (*r*=3) (Fig. 2) as it displays a sharp peak 3.5*kHz.* Its reflection coefficients at the problematic frequency are +1 and -1 one after the other (steady phase change of *π*). However, while this avoids a simple tilted reflected lobe, the prediction from the Fourier method is unlikely to be accurate, as mutual interactions will tend to 'smooth out' the surface pressure distribution. The sequences that have *r*=2 and *r*=4 display more unsteady diffusion with frequency and display many dips in the diffusion coefficient graph.

B. Power Residue Diffusers

Power Residue diffusers (PWRD) of the same family can be separated into pairs that diffuse similarly. Each diffuser is paired with its inverse, which is found in the same family *M*/2 sequences away ($|r_1 - r_2| = M/2$). The PWRDs that display period *N*=9 are the most promising ones as they are short enough for easy application while being long enough to be considered as a random sequence of wells. Such diffusers can be generated for primes *p*=19 (*M*=2), *p*=37 (*M*=4) and *p*=73 (*M*=8).

The first estimations show that PWRDs do meet the initial goal of postponing the flat plate frequency while displaying stable diffusion with frequency. Further calculations need to be made in order to conclude on the performance of PWRDs.

MODULATION AND PERIODICITY

In order to consider large areas covered with diffusers more than one period per structure was used for the predictions. The repetition of the sequences introduces the problem of periodicity. Periodicity causes harmonics to be created in the autocorrelation function of each structure. This creates sharper lobes and as a result less uniform scattered distribution[5, 8]. A method of dealing with the phenomenon of periodicity is to modulate the diffuser with another using a binary pseudorandom sequence. This sequence sets the order of the diffusers, with 1 corresponding to the first diffuser and 0 to the second.

As shown above, at some frequencies, LSDs simply redirect the sound because they act like beam steerers. In general, diffusers should be dispersing sound instead of redirecting it. An effective solution is to modulate the diffuser with a sequence that at the problematic frequencies redirects sounds into another angle. Such a sequence can be the inverse of the first sequence[8] or another LSD from the same family (Fig. 3). The sole major lobe of the periodic diffuser has been substituted by two of less energy. Thus, the incident wave is scattered more uniformly. Using the inverse of a diffuser to modulate it has an advantage over using two separate diffusers as the inverse of a diffuser is the diffuser turned upside down. Modulated in this manner the overall performance of diffuser is improved as shown in Fig. 4.

The LSDs of this family that have *r*=2 and *r*=4 will improve their performance with modulation but they will not display as a stable a diffusion coefficient with frequency as the *r*=1 case.

Another type of modulations that can be made is to modulate one sequence with its mirror sequence. A mirror of the sequence can be created by inversing the order of the sequences coefficient. Since the LSD of period *N*=6 and *r*=1 is [6,25,26,15,10,23] its mirror will be [23,10,15,26,25,6]. This modulation has the same advantage with the inverse one as mirrors diffuser are the diffusers themselves flipped from left to right. Such a modulation scatters a bit worse than the modulations with the inverse.

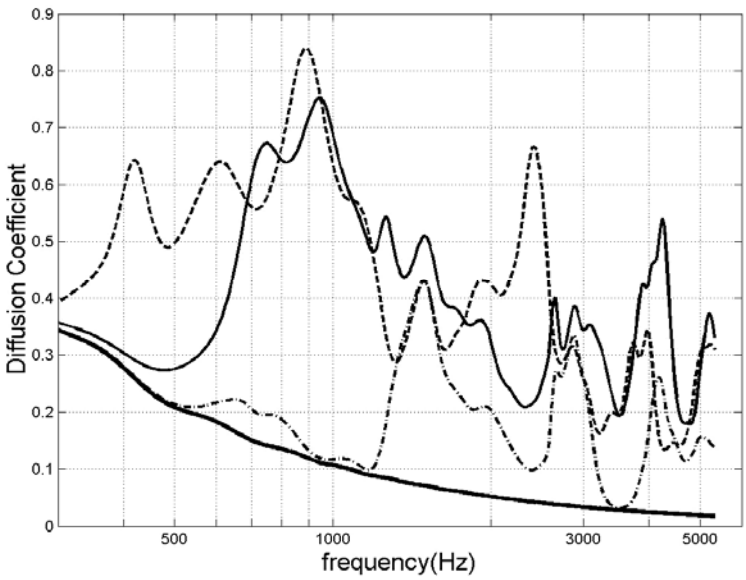

Figure 4: Prediction of the diffusion coefficient using the Fourier Method of LSD diffuser (*p*=42, *r*=1) (design frequency 500*Hz*) periodic -·-·-·-·-, modulated using the binary sequence [1,0,0,1,0,1] with its inverse ———, and with another LSD of the same family (*r*=5) ------ and plane surface of equal length ▬▬▬.

CONCLUSIONS

This paper has suggested the use of Type-II Lüke and Power Residue sequences in designing number theoretic diffusers. The logical and mathematical background has been presented and their performance has been. Classic Schroeder diffusers based on quadratic residue and primitive root sequences suffer from flat plate effect where no scattering is achieved. Type II Lüke and Power Residue sequences, use larger integer-bases sequences, and consequently the diffusers' flat plate frequencies are at much higher frequencies. The results show that Type-II Lüke diffusers perform like beam steerers at some frequencies, and consequently at these frequencies dispersion is poor. Modulation techniques have been presented to mitigate this problem. First estimations of the performance of Power Residue diffusers have shown them to be more promising as they do not seem to display any major problems until their flat plat frequency, which is much higher. Further calculations need to be made in order to conclude on the performance of Power Residue diffusers.

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