Extension of the MOT model for radial modes of piezoelectric ceramics exhibiting Bessel polarizations

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Abstract

The standard procedure describing the radial vibration modes of piezoceramic discs assumes homogeneous polarization. Nevertheless, the fact that non-uniform polarization can affect dramatically the vibration modes of piezomaterials has motivated its application to radiation and frequency control. In this work, with the purpose of studying the radial modes of a non-homogeneously polarized disc, an extension of the Meitzler-O’Brayan-Tiersten (MOT) model, which is based on the proposal a correspondence between the piezoelectric constants and the radius-dependent polarization, is presented. The extended model is applied to a piezoelectric disc polarized with a zeroth-order first-kind Bessel function. Following the MOT methodology and because of the selected polarization, an electromechanical coupling factor depending on frequency and polarization is defined. The extended model is analyzed numerically in order to determine the influence of polarization on the electrical response of the vibration modes, and is applied to experimental Bessel transducers for explaining the modal behavior of these devices.

I. INTRODUCTION

Nowadays, piezoelectric transducers are widely used in electromechanical sensors, actuators, transformers, non-destructive testing devices and electro-optical modulators, by exploiting the piezoelectric property that couples the electrical and mechanical responses. In order to carry out the design process of this kind of devices, it is important to identify their vibrational characteristics, which can be determined by the Maxwell equations and the piezoelectric constitutive equations [1]. Despite the simplicity of the geometry considered here, the vibration of a piezoelectric device is a complex process and it is usually difficult to obtain analytical solutions. The general approaches appearing in the literature are the use of the Finite Elements Method (FEM) [2][3] and the development of two- or three-dimensional analytical models [4-6]. Though FEM-based algorithms are capable to solve this problem, theoretical methods are always desirable because numerical approaches do not give sufficient understanding of the physical phenomenon.

Homogeneously polarized piezoelectric discs are the simplest and commonest configurations for a transducer, so its vibration characteristics are usually required in transducer design and applications. There are different types of vibration modes for this geometry and each one has a unique distribution of displacements. Based on one-dimensional approaches, the modes are classified in pure and coupled modes. Pure modes are called radial and thickness modes. The Meitzler-O’Brayan-Tiersten (MOT) model [4], presented in the IEEE Standards on Piezoelectricity [7], is the classical model for the radial modes of uniformly polarized discs.

On the other hand, non-homogenously polarized piezoelectric materials have been used to control the vibration and radiation of transducers [8-10]. In a recent report [10], we presented that a piezoelectric transducer with a Bessel vibration profile can be obtained by direct polarization. There, piezoceramic discs with spatial discretizations of Bessel functions are driven by a time-dependent uniform field and their radiations and vibration modes are studied. In the mentioned work, suppression of some radial
modes was found. This atypical behavior led our interest to the study of the radial modes of non-uniformly polarized discs.

In the present work, we extend the MOT model to non-homogeneously polarized discs and the radial vibration characteristics of a disc with Bessel polarization are studied. It is found that, for Bessel polarizations with zero value at the boundary of the disc, all radial modes except one are suppressed. This behavior is similar to modal transducers [11-13] and can be applied in frequency control as an alternative to electrode distribution or in combination with it. In order to validate the model, theoretical and experimental results are compared for three Bessel transducers [14].

The work is organized in the following way. In Section II, a theoretical model for radial modes in a piezoelectric disc is described. In Section III, the hypothesis of non-homogeneous polarization is introduced. In Section IV, polarization is assumed as a first-kind zeroth-order radial Bessel function, the differential equation is solved and the electrical impedance is found. In Section V, general results of the model are presented and results obtained for three piezoelectric Bessel transducers are shown and compared with experiments.

II. NON-HOMOGENEOUS POLARIZATION

We consider that the polarization depends on the radius \( r \). The piezoelectric constants will also depend on the radius and it is assumed that they have the following form

\[
e_{31}(r) = e_{31}P(r) \\
e_{13}(r) = e_{13}P(r) \\
e_{33}(r) = e_{33}P(r)
\]

where \( e_{ij} \) is the value of the piezoelectric constant when the ceramic is it polarizes up to the saturation value. Function \( P(r) \) is the percent of polarization in function of radius, given by

\[
P(r) = \frac{p(r)}{p_0}
\]

where \( p(r) \) is the polarization in function of radius \( r \) and \( p_0 \) is the value of the remainder polarization when the material is polarized up to saturation.

Assuming metallic electrodes in the main faces and harmonic voltage excitation \( V = V_0 e^{i\omega t} \), and by substituting equations (1), in the motion equation [1] we obtain:

\[
\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r} \frac{\partial u_r}{\partial r} + \left[ \frac{\omega^2}{c_{ii}/\rho} - \frac{1}{r^2} \right] u_r + \frac{\partial P e_{33}^p}{\partial r} \frac{V}{2b} = 0.
\]

(3)

\( J_1 \) is the first-kind first-order Bessel function, \( Y_1 \) is the second-kind first-order Bessel function.

III BESSEL ZERO POLARIZATION

The Bessel transducer appear in the literature because its can be radiate a limit diffraction beam. Some authors construct this devices using non homogeneously polarization in radial direction [8][10]. In this section we will be analyze the influence of this kind of polarizations in radial vibration. Now we assume that the polarizations follow a first kind and cero order Bessel function:

\[
P(r) = J_0 (\alpha r)
\]

(4)

were parameter \( \alpha \) governs the behavior of polarization functions, and

\[
\frac{\partial P}{\partial r} = -\alpha J_1 (\alpha r)
\]

(5)

For this particular polarization equation, (3) take the form
\[\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r} \frac{\partial u_r}{\partial r} + \left[ \frac{\omega^2}{\left( \frac{c_{i1}^p}{\rho} \right)^2} - \frac{1}{r^2} \right] u_r = -\alpha J_1 (\alpha r) \frac{e^{0_1} V}{c_{i1}^p 2b} = 0 \quad (6)\]

for equation (6) is possible to find an analytical solution and having in cont that function \(Y_1\) is singular in \(r = 0\) solution for (6) take the following form where

\[\alpha = \frac{\omega}{\sqrt{\frac{c_{i1}^p}{\rho}}} , \quad (7)\]

\[u_r(r) = A_r J_1 (\alpha r) - \frac{e^{0_1} V}{c_{i1}^p 2b} \frac{\alpha J_1 (\alpha r)}{\alpha^2 - \alpha^2} , \quad (8)\]

Equation (8) is subject to boundary condition

\[T_r \mid_{r=a} = 0 \quad (9)\]

Evaluation (8) in (9) constant \(A\) can be expresses as:

\[A = -\frac{e^{0_1} V}{c_{i1}^p 2b} \left\{ \frac{\alpha J_0 (\alpha a) + (\sigma^p - 1) \frac{J_1 (\alpha a)}{a}}{\alpha^2 - \alpha^2} \left[ \frac{\alpha J_0 (\alpha a) + (\sigma^p - 1) \frac{J_1 (\alpha a)}{a}}{\alpha^2 - \alpha^2} \right] \right\} , \quad (10)\]

The equation for current in the disc is obtained deriving the charge respect to the time

\[I = \frac{dQ}{dt} = -2\pi i \omega \int D_r \rho dr , \quad (11)\]

Substitution of (8) and in (11) and divided by the voltage, the admittance of system is obtained as follow form

\[\frac{1}{Z} = \frac{I}{V_0} = \frac{i \omega \pi c_{i1}^p}{b} \left\{ \frac{e^{0_1}^2}{c_{i1}^p e_{33}^p} \left[ \alpha A_N I_1 - \frac{\alpha}{\alpha^2 - \alpha^2} I_2 \right] - \left[ \frac{\alpha^2 e_{33}^p}{e_{33}^p} + I_2 \left( 1 - \frac{e_{33}^p}{e_{33}^p} \right) \right] \right\} , \quad (12)\]

where

\[I_1 = \int_0^a J_0 (\alpha r) J_0 (\alpha r) r dr = \frac{a}{\alpha^2 - \alpha^2} \left[ \alpha J_0 (\alpha a) J_1 (\alpha a) - \alpha J_1 (\alpha a) J_0 (\alpha a) \right] \quad (13)\]

\[I_2 = \int_0^a \left( J_0 (\alpha r)^2 \right) r dr = \frac{a^2}{2} \left( J_0 (\alpha a)^2 \right) \quad (14)\]

and

\[A_N = \frac{\alpha}{\alpha^2 - \alpha^2} \left[ \alpha J_0 (\alpha a) + (\sigma^p - 1) \frac{J_1 (\alpha a)}{a} \right] + J_0 (\alpha a) \left( \alpha^2 - \alpha^2 \right) \quad (15)\]

\[A_D = \left( \alpha^2 - \alpha^2 \right) \left[ \alpha J_0 (\alpha a) + (\sigma^p - 1) \frac{J_1 (\alpha a)}{a} \right] \quad (16)\]

At the electrical resonance frequency, the current goes to infinite. Analyze equation (12) it is possible determined this condition and can be writing as:
\[
\left( a \alpha \frac{J_0(\alpha a)}{J_1(\alpha a)} + (\sigma^p - 1) \right) = 0
\]  
(17)

Eq. (17) is the same expression for resonance in homogeneously polarized disc and coincides with mechanical resonance [4]. The antiresonance frequency result when the current goes to cero. Applying this condition to Eq. (12) and make an algebraic manipulation the resulting equation is

\[
a \alpha \frac{J_0(\alpha a)}{J_1(\alpha a)} = 1 - \sigma^p - 2 \left( k^p \right)^2
\]  
(18)

where

\[
\left( k^p \right)^2 = \frac{-\alpha a A_N I_1(k^p)^2 (J_1(\alpha a))^{-1}}{a^2 \varepsilon_{31}^p + 2 \left( 1 - \frac{\partial}{\partial^2 - \alpha^2} \right) I_2}
\]  
(19)

IV.2 BESSEL TRANSDUCER

The Bessel transducers, first presented in the literature by Hsu [3], are constructed taking a Bessel function with a zero in the border of disc. Following this assumption, \( \alpha \) satisfies the relation

\[
\alpha = \frac{x_N}{a}
\]  
(20)

where \( x_N \) is the \( N \)-th zero of the Bessel function. In order to prove the hypothesis presented in the previous section, the theoretical results are compared to our previous experiments [paper experimental sometido] concerning three transducers with Bessel polarizations constructed using ceramic (PZT 53/47 + 1wt% Nb) discs with thickness 1mm and diameter 24 mm. These transducers were called M2, M3 and M4 and conceived for exciting the second, third and fourth radial modes, respectively. In Fig. 4, the schematics of the associated theoretical polarization functions in comparison with the actual discretized polarizations applied to the discs, are shown. The influence of distinction between theoretical and experimental polarizations will be discussed later.
The extended model proposed in this work predicts the suppression of every mode except one in each Bessel transducer, which is confirmed by the mentioned experiments. In Fig. 5, comparisons between impedances of the classic disc and transducers M2, M3 and M4 with theoretical results (Eq. (12)) are shown. Note that all radial modes in M2, M3 and M4 are suppressed except the target modes 2, 3 and 4, respectively, which is the typical behavior of modal transducers [8-9]. The theoretical and experimental results concerning impedances evidence good agreement. The model proposed here accurately predicts both suppressed and reinforced modes. The slight differences between theory and experiments might be attributed to a variety of factors, being two of them the most important. The first factor is the influence of thickness, as the model is consistent with thin discs, which causes the small shifting of the resonance and antiresonance frequencies. The second factor is that the theoretical model uses the continuous polarization function while in experiments the polarization is a step-like discretization of the theoretical one (see Fig. 1). The influence of this second factor is visible in the difference in capacities between theoretical and experimental curves, being more evident for M2 transducer as it is the one having the largest inactive area. Suppression of modes can be understood in the sense that deformations of discs must satisfy the boundary conditions of the new systems M2, M3 and M4. Moreover, a constant electric field inside the material, which is induced by the metallic electrodes when the voltage source is connected, is only possible in modes for which deformation behaves similar to the polarization function, as it is well known that in a piezoelectric system not every mechanical mode can be electrically excited. As it is shown in Fig. 5, in uniformly-polarized discs all radial modes are coupled to the electrical part of the system, while in Bessel-polarized discs not every mode is electrically coupled.
V. CONCLUSIONS

In this paper, with the objective of explaining theoretically the electrical behavior of experimental Bessel transducers, an extension of the MOT model is proposed. This model includes the influence of non-uniform polarization on the piezoelectric constants. As a result of the adopted methodology and the particular polarization, a new electromechanical coupling factor depending on the frequency and the polarization is defined. The defined coupling factor reduces to the standard one in the uniform polarization limit and it can be used as a merit figure in order to analyze the suppression of modes in modal transducers. The technique proposed here is applied to three built experimental Bessel transducers and theoretical and experimental results evidence good agreement.

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REFERENCE


Fig.2. Comparison of theoretical and experimental results for electrical impedance of uniform polarized transducers a) and for Bessel transducers b) M2, c) M3 and d) M4