A Comprehensive Model for the Turbulent Boundary Layer Pressure Spectrum

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ABSTRACT
Using test data from wind tunnel and water channel fluid flow tests, the model originally developed by Chase for the low wavenumber spectrum of the pressure in a turbulent boundary layer has been extended to cover the entire wavenumber and frequency range. Particular attention is given to the low wavenumber and high frequency regions. The low wavenumber spectrum level is found to be nearly constant and a theoretical basis for this is proposed. The high frequency spectrum level is found to correlate best with the inner variables (shear stress and viscosity) while the remaining regions correlate best with the outer variables (convection velocity and boundary layer thickness).

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1. INTRODUCTION
The pressure fluctuations in a turbulent boundary layer (TBL) flow over a surface are important sources of structural vibration and transmitted sound. The strength of the excitation depends on both the spatial and temporal matching of the pressure and vibration fields. Typically this is described by functions of wavenumber, \( k \), and frequency, \( \omega \), which are Fourier transforms of the spatial and temporal fields, respectively. A typical example of an empirical formula for the pressure spectrum, \( \Phi_p \), describing the pressure field in a fully developed TBL is given by the basic Chase model [1]

\[
\Phi_p(k, \omega) = C \rho_i^2 u_c^3 k^2 \left[ k^2 + (2\pi)^2 (\omega \delta u_c - k \delta)^2 + \delta^{-2} \right]^{5/2}
\]  

where \( \rho_i \) is the ambient fluid density, \( u_c \) is the convection velocity of the turbulence, \( \delta \) is the effective thickness of the turbulent boundary layer, and \( C \) is a constant.

Of particular interest in the development of Equation 1 was the low wavenumber component of \( \Phi_p \) which plays an important role in the structural response to TBL pressures in some cases. This is illustrated in Figure 1 where typical one dimensional wavenumber spectra of a TBL pressure field (at Mach 0.1) and a finite flat panel vibration response function are compared for a frequency of 500 Hz.

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The panel vibration is determined by the product of these two spectra which is dominated by two wavenumber regions: near the TBL convection peak and near the panel response peak. The latter is considered to be in the low wavenumber region of the TBL spectrum. Also shown is a typical spectrum of the acoustic pressure present which is usually much lower in amplitude than the TBL pressure but can also contribute to the panel vibration.

Measurements of the low wavenumber TBL pressure [2, 3, 4] generally have exceeded the levels predicted by Equation 1. Many attempts have been made to account for this difference [4, 5, 6]. The first part of this paper presents the results of new measurements from wind tunnel and closed water channel tests and proposes a theoretical basis for a modification to the basic Chase model.

The single point frequency spectrum of the TBL pressure, \( \Phi_p(\omega) \), predicted by the basic Chase model is obtained by integrating Equation 1 over wavenumbers, resulting in

\[
\Phi_p(\omega) = C' \left( \rho_o u_c \omega \right)^2 \left[ 1 + \left( \frac{u_c}{l} \frac{\omega}{\delta} \right)^2 \right]^{3/2} / \omega
\]  

(2)

Figure 2 compares this with typical measured results in a wind tunnel. It can be seen that the data falls off faster at high frequencies than Equation 2 predicts. Many attempts have been made to account for this difference [3, 6, 7, 8]. The second part of this paper presents the results of new measurements from wind tunnel and closed water channel tests and proposes a theoretical basis for a modification to the basic Chase model.
2. LOW WAVENUMBER TBL PRESSURE MEASUREMENTS AND ANALYSIS

The test results reported here use the method of Martin [2] to determine the low wavenumber component of the TBL pressure spectrum. The vibration response of a flat plate exposed to a TBL flow is used to back-calculated the TBL pressure levels at wavenumbers matching those of the bending vibration modes of the plate. Wind tunnel tests previously reported [4, 9, 10] have been extended to include more flat plate configurations to determine the low wavenumber component of the TBL pressure spectrum. The Calvin Engineering Department wind tunnel configuration is shown in Figure 3. Tests have been performed with variety of flat, rectangular plates constructed of steel, aluminum, glass, and polycarbonate, measuring 0.58 m by 0.18 m, with thicknesses varying from 0.61 mm to 4.0 mm, and positioned in the 3.0 m long test section at 0.9 m and 2.1 m downstream from the air flow inlet.

Figure 2. TBL pressure power spectral density (PSD) at 2.0 m with 20 m/s air flow

Figure 3. Calvin Engineering Department wind tunnel tests
The plates are supported around the perimeter with 25 x 50 mm steel beams, as shown in Figure 4, to approximate clamped edges. This minimizes the amplitude of the side bands in the wavenumber transform of the plate vibration mode shape [2] and makes them less susceptible to excitations at high wavenumbers. An example of a plate mode shape and corresponding two dimensional wavenumber spectrum, \( \Psi(k_x, \omega) \), is shown in Figure 5. The mode shape is measured with a scanning laser vibrometer and acoustic excitation.

\[ \rho_s h_s \eta_s S_{\omega}(\omega) = \int \Phi_p(k, \omega) \Psi(k, \omega) \, dk \]  

(3)

where \( \rho_s \), \( h_s \), \( \eta_s \) are the plate density, thickness and damping, respectively.

Figure 4. Typical test plate mounting with perimeter support beams

Figure 5. Measured mode shape and wavenumber spectrum of (5,1) mode of 3.5 mm glass panel at 1142 Hz
With a plate installed in the wind tunnel flush to the test section wall (Figure 3) the acceleration spectrum is measured with a scanning laser vibrometer and the TBL pressure spectrum is back calculated using Equation 3 for each vibration mode. Figure 6 shows an example of a measured acceleration spectrum. There is enough frequency separation between most of the modes from 200 to 2000 Hz to determine their individual response levels. The damping loss values typically vary from 0.01 at lower frequencies to 0.002 at higher frequencies.

![Figure 6. Measured power spectral density (PSD) of 0.61 mm steel plate acceleration level with 20 m/s air flow (0.625 Hz FFT bandwidth)](image)

Even though the acoustic pressure is small compared to the turbulent pressure it can still contaminate the plate vibration measurement. This is because the acoustic pressure is concentrated at wavenumbers around $k_o$ while the turbulent pressure is spread over a wavenumber range of about $k_c$. Previous tests in the wind tunnel [4] have shown the TBL/acoustic pressure ratio to be about 25 dB, and at 25 m/s the ratio of $(k_c/k_o)^2$ is 26 dB. To remove the acoustic contamination a method similar to Bonness [3] is used. A pressure transducer outside the TBL is used to measure the amount of plate vibration correlated with the acoustic pressure and this is removed from the measured vibration. Figure 7 shows an example of this. This usually only alters the data for the first few vibration modes at low frequencies.

![Figure 7. Measured and corrected PSD of 0.81 mm aluminum plate acceleration level with 25 m/s air flow.](image)
New tests have also been performed using a closed water channel shown in Figure 8. The 3.0 m long test section consists of a lower half pipe (nominally 0.25 m diameter, schedule 80 PVC) with test plates rigidly attached to the top and supported by 38 mm by 50 mm beams around the perimeter. A variety of flat, rectangular plates constructed of steel, aluminum and brass, measuring 0.69 m by 0.15 m, with thicknesses of 0.81 mm to 3.2 mm, and positioned in the test section at 0.9 m and 2.1 m downstream from the water flow inlet have been tested. The supply tank provides gravity fed water from a height of about \( h = 2.5 \) m giving a theoretical maximum velocity of \( \sqrt{2gh} = 7.0 \) m/s. However, due to losses in the settling tank, the tests achieved a maximum velocity of 5.3 m/s. Lower speeds are achieved by partially closing the exit valves. The cross-sectional area of the lower half pipe increases by 20% along the length in order to account for the growth of the boundary layer at the wall and to achieve zero pressure gradient.

The water in the return pump section is isolated from the test water by using free, open flow in both the supply and discharge tanks. The pumps are enclosed in a noise and vibration isolation box. The 0.6 m x 0.6 m x 2.0 m settling tank is lined with crushed cell foam and steel mesh in order to dampen the acoustic waves in the test section. Also included is a flow straightening section and a 16:1 contraction section. Wire screen is used in all tanks to reduce the turbulence and eliminate entrained air bubbles. The vibration of the pipe sections is suppressed by surrounding them with sand.

The acceleration spectrum is measured with a scanning laser vibrometer and the TBL pressure spectrum is back calculated using Equation 3 for each vibration mode. Figure 9 shows an example of a measured acceleration spectrum. There is enough frequency separation between most of the modes from 50 to 1500 Hz to determine their individual response levels. The damping loss values typically vary from 0.02 at lower frequencies to 0.005 at higher frequencies. The surface mass includes fluid loading in this case.

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*Figure 8. Water channel system using a closed top half-pipe*
In this paper only the data obtained for purely axial wavenumbers are presented so \( k = k_c \) and only the \((m,1)\) plate modes are considered. The TBL pressure spectrum is usually presented in non-dimensional form, but there are different forms of this in the literature. One form uses the “outer variables,” the convection velocity, \( u_c \), and the TBL displacement thickness, \( \delta^* = \delta/8 \). Focusing on the low wavenumber region \((k_x \ll \omega/u_c)\), Equation 1 becomes (also with \( \omega u_c \gg \delta^* \))

\[
\Phi_p(k_x, \omega) \frac{u_c}{l} \left[ \delta^* (\rho_o u_c^2) \right] \sim C \left( \frac{\omega}{k_c} \right)^5 \left( \frac{k_x}{k_c} \right)^2 / (\omega \delta^* u_c)^3
\]

where \( k_c = \omega / u_c \).

Figure 10 presents the complete set of data for this condition with the normalized pressure spectrum plotted against the normalized axial wavenumber for different values of \( \omega \delta^*/u_c \).

The data show no strong dependence on \( k \) for \( k << k_c \) where as Equation 4 predicts a \( k^2 \) dependence. This is consistent with other measurements [2, 3]. A theorem by Kraichnan [11] is often sited as the reason for the \( k^2 \) dependence in theoretical results. This is derived for a uniform boundary layer over an infinite flat surface with a constant...
thickness and mean flow parallel to the surface. Then there can be no spatially uniform \((k_x = 0)\), non-zero pressure against the surface. However, in reality the boundary layer is growing and new kinetic energy is entering the boundary layer at a nearly uniform rate spatially. This is illustrated in Figure 11.

![Figure 11. Illustration of growth of TBL and addition of kinetic energy, \(\frac{1}{2} \rho_o u_c^2\)](image)

An approximation for the spatially uniform pressure is

\[ p \sim \rho_o u_c^2 \frac{d\delta}{dx} \]  

(5)

and the growth of the boundary layer profile is approximated by

\[ \delta = 0.16 x \text{Re}^{-1/7} = 0.16 x^{6/7} (\nu/U)^{1/7} \]

(6)

where the Reynolds number is \( \text{Re} = U x / \nu \), \( U \) is the free stream velocity and \( \nu \) the fluid kinematic viscosity.

Then

\[ p \sim \rho_o u_c^2 (\text{Re})^{-1/7} \]

(7)

which has a very weak dependence on \( x \). Approximating the Fourier transforms of \( p^2 \) gives

\[ \Phi_p(k, \omega) \sim (\rho_o u_c^2)^2 / \omega k_c^2 \]

(8)

which can be included in Equation 1 by adding a term to the numerator proportional to \( k_c^2 \). By curve fitting this to the data in Figure 10 a scaling of \( 1/\pi \) is determined. The revised model is then

\[ \Phi_p(k, \omega) = C \rho_o^2 u_c^3 (k^2 + k_c^2 / \pi) [k^2 + (2\pi)^2 (\omega u_c - k_x)^2 + \delta^2]^{-5/2} \]

(9)

which is also plotted in Figure 10.

3. HIGH FREQUENCY TBL PRESSURE MEASUREMENTS AND ANALYSIS

Figure 2 illustrates the inadequacy of the basic Chase model for the TBL pressure spectrum at high frequencies. Most attempts to fix this \([3, 6, 7, 8]\) have added an empirically derived term to roll off the model at high frequencies. One of the difficulties in obtaining reliable high frequency data is the reduced sensitivity of a microphone surface when the diameter, \( d \), is a significant fraction of the turbulent eddy size, \( k_c d \geq 1 \). Up to \( k_c d = 2\pi \) the reduced sensitivity can be corrected \([12]\). In the measurements reported here the microphones are mounted behind a pin hole with \( d = 0.8 \text{ mm} \) and all the data have been corrected for its response function.
The appropriate variables for the high frequency scaling can be found by looking at the measured pressure spectra using different scaling parameters. Figure 12 shows an example of the measured TBL pressure spectra at one location and four flow velocities scaled by the outer variables, $u_c$ and $\delta^*$.

![Figure 12](image1.png)

**Figure 12.** Measured TBL pressure spectra at 1.8 m position in wind tunnel with four air flow speeds, normalized by the outer variables ($q = \frac{1}{2} \rho U^2$)

The normalization looks good except for the fact that $\delta^*$ is not varying. Figure 13 shows an example of the measured TBL pressure spectra at four locations and one flow velocity scaled by the outer variables. Here the frequency scaling does not collapse the data at high frequencies.

![Figure 13](image2.png)

**Figure 13.** Measured TBL pressure spectra at four positions in wind tunnel with 30 m/s air flow speed, normalized by the outer variables ($q = \frac{1}{2} \rho U^2$)

Another choice of scaling parameters that has been found to work at high frequency uses the inner variables; fluid kinematic viscosity, $\nu$, and surface shear stress, $\tau_w = \rho o u_f^2$, or just $u_f$ the friction velocity [3, 6, 7, 12]. Figure 14 shows a set of measured TBL pressure in both air and water normalized by these variables. The high frequency data collapse but the low frequency data do not.
This indicates that \( \nu/\nu_t \) is the length scale for the high frequency normalization. In a TBL the inner most shear layer at the surface is usually determined to have a thickness of about \( 6 \nu/\nu_t \). In the development by Chase [1] of Equation 1, the last step uses an integral of the form

\[
\Phi_p(k, \omega) = C' \rho_o^2 u_c^3 k^2 \int \left[ k_y^2 + k^2 + (2\pi)^2 (\omega u_c - k_x)^2 + \delta^2 \right]^3 d k_y
\]  

Neglecting the effects of the shear layer this integral is evaluated to infinity for \( k_y \), the wavenumber perpendicular to the surface. However, as Chase suggests, one of the effects of the shear layer is to suppress the small scale turbulent eddies and reduce the high frequency pressure spectrum. To approximate this effect the integral in Equation 10 can be evaluated to limiting values of \( +/\nu_t/6 \nu \). In doing so a factor of \( [1 + k^2 (6 \nu/\nu_t)^2]^{-2} \) is appended to Equation 1.

Integrating this over wavenumber to obtain the frequency spectrum gives

\[
\Phi_p(\omega) = C' (\rho_o u_c^2)^2 /\left\{ \omega \left[ 1 + (u_c / \delta \omega)^2 \right]^{3/2} + [6 \omega \nu/\nu_t^2] \right\}^{1/2}
\]  

This revised Chase model is plotted along with a TBL pressure measurement in water in Figure 15 showing excellent agreement.
4. CONCLUSIONS
Measurements of the TBL pressure and plate vibration in both wind tunnel and closed water channel have been used to extend the basic Chase model to more accurately represent the low wavenumber and high frequency regions. Theoretical justifications for these modifications are used to infer the best scaling parameters to be used.

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6. REFERENCES