ON THE STABILITY OF THE MOTION FOR A NEO-HOOKEAN SYSTEM

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Abstract
It is well known that the vibrations are a major cause for the instability in the mechanical systems and major source of noises. In this paper we propose a simple system with two degrees of freedom based on a non-linear elastic element. For this system, it is proved in the present paper that the motion is stable, but not asymptotically stable. A comparison between the non-linear case and the linear case is performed, and for the both cases the eigenpulsations are also determined. All theoretical results are validated by numerical simulation.

Keywords: neo-Hookean, motion, stability, numerical validation.

1. Introduction
The system purposed for the study is described in figure 1, a. It consists of the masses \( m_1 \) and \( m_2 \) linked one to another by the linear spring of stiffness \( k \). The mass \( m_1 \) can be considered to be the foundation of the machine-tool, and the mass \( m_2 \) the machine-tool itself. The mass \( m_2 \) is linked to the ground by the non-linear spring 1 for which the elastic force writes

\[
F = k_1 x - \frac{\epsilon_1}{x^2},
\]

where \( x \) is the elongation of the spring.
The fundamental working hypothesis is that

\[
\epsilon_1 > 0.
\]

The system has two degrees of freedom, that is the displacements \( z_1 \) and \( z_2 \) of the two masses in the vertical direction.

2. The equations of motion
Isolating the two masses (fig. 1, b), one obtains the differential equations of the motion
where $g$ is the gravitational acceleration.

Denoting

$$\xi_1 = z_1, \quad \xi_2 = z_2, \quad \xi_3 = \dot{z}_1, \quad \xi_4 = \dot{z}_2,$$

the relations (3) transform in a system of four first order non-linear differential equations,

$$\dot{\xi}_1 = \xi_3, \quad \dot{\xi}_2 = \xi_4, \quad \dot{\xi}_3 = \frac{1}{m_1} \left[ -k_1 \xi_1 + \frac{e_1}{\xi_1^2} + k(\xi_2 - \xi_1) + m_1 g \right], \quad \dot{\xi}_4 = \frac{1}{m_2} \left[ m_2 g - k(\xi_2 - \xi_1) \right].$$

3. The equilibrium positions

These positions found at the intersections of the nullclines, so that one obtains the system

$$\xi_3 = 0, \quad \xi_4 = 0, \quad -k_1 \xi_1 + \frac{e_1}{\xi_1^2} + k(\xi_2 - \xi_1) + m_1 g = 0, \quad m_2 g - k(\xi_2 - \xi_1) = 0.$$  (6)

Summing the last two relations (6), it results

$$-k_1 \xi_1 + \frac{e_1}{\xi_1^2} + (m_1 + m_2)g = 0,$$

wherefrom

$$k_1 \xi_1^3 - (m_1 + m_2)g \xi_1^2 - e_1 = 0.$$  (8)

In the sequence of the coefficients of powers of $\xi_1$ in the relation (8) there exists only one variation of sign such that applying the Descartes theorem one deduces that the equation (8) has only one positive real root. Making now $\xi_1 \mapsto -\xi_1$, one obtains the equation
\[ k_1 \xi_1^3 + (m_1 + m_2)g \xi_1^2 + e_1 = 0 \]  

(9)

for which there exists no variation of sigh in the sequence of the coefficients so that the Descartes theorem assures us that we have no negative real root for the equation (8). In conclusion, the equation (8) has exactly one positive real root, name it \( \xi_1 \).

The last relation (6) becomes now a linear equation in the unknown \( \xi_2 \) and therefore it has only one solution,

\[ \xi_2 = \frac{m_2 g}{k} + \xi_1. \]  

(10)

We proved in this way that the system has only one equilibrium position \((\overline{\xi}_1, \xi_2, 0, 0)\).

Let us denote by \( f(\xi_1) \) the function \( f : \mathbb{R} \rightarrow \mathbb{R} \),

\[ f(\xi_1) = k_1 \xi_1^3 - (m_1 + m_2)g \xi_1^2 - e_1 \]  

(11)

for which the derivative is

\[ f'(\xi_1) = 3k_1 \xi_1^2 - 2(m_1 + m_2)g \xi_1. \]  

(12)

The equation \( f'(\xi_1) = 0 \) has the solutions

\[ \xi_1^{(1)} = 0, \quad \xi_1^{(2)} = \frac{2(m_1 + m_2)g}{3k_1}, \]  

(13)

\( \xi_1^{(1)} \) being a point of maximum, and \( \xi_1^{(2)} \) a point of minimum. In addition,

\[ f(0) = -e_1 < 0. \]  

(14)

Graphically, the situation is presented in figure 2. It follows from here that

\[ \overline{\xi}_1 > \frac{2(m_1 + m_2)g}{3k_1}. \]  

(15)
4. The stability of the equilibrium

Let us denote by $f_k$, $k = 1, 4$ the right-hand terms of the relations (5) and by $j_{kl}$ the partial derivatives

$$j_{kl} = \frac{\partial f_k}{\partial x_l}, \quad k = 1, 4, \quad l = 1, 4.$$  

We have

$$j_{31} = 0, \quad j_{32} = 0, \quad j_{13} = 1, \quad j_{14} = 0,$$

$$j_{21} = 0, \quad j_{22} = 0, \quad j_{23} = 0, \quad j_{24} = 1,$$

$$j_{31} = \frac{1}{m_1} \left( -k_1 - k - \frac{2v_1}{\xi_1} \right), \quad j_{32} = \frac{k}{m_1}, \quad j_{33} = 0, \quad j_{34} = 0,$$

$$j_{41} = \frac{k}{m_2}, \quad j_{42} = -\frac{k}{m_2}, \quad j_{43} = 0, \quad j_{44} = 0.$$  

The characteristic equation

$$\det(J - \lambda I) = 0$$  

in which $J$ is the Jacobi matrix, $J = [j_{kl}]_{k,l=1,4}$ and $I$ is the fourth order unity matrix, reads

$$\begin{vmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ j_{31} & j_{32} & -\lambda & 0 \\ j_{41} & j_{42} & 0 & -\lambda \end{vmatrix} = 0.$$  

Multiplying the third column by $\lambda$, and adding it to the first column, the fourth column by $\lambda$ and adding it to the second column, results the equation

$$\lambda^4 - (j_{31} + j_{42})\lambda^2 + j_{31}j_{42} - j_{32}j_{41} = 0.$$  

From the Routh–Hurwitz criterion we deduce that the equation has not all the roots with negative real part and therefore the equilibrium can not be asymptotically stable.

On the other hand, the roots of the equation (23) are

$$\lambda^2 = \frac{j_{31} + j_{42} \pm \sqrt{(j_{31} - j_{42})^2 + 4j_{32}j_{41}}}{2}.$$  

Keeping into account the expressions (17)-(20), we have

$$j_{31} + j_{42} = \frac{k_1 + k + \frac{2v_1}{\xi_1}}{m_1} - \frac{k}{m_2} < 0,$$

$$4j_{32}j_{41} = \frac{4k^2}{m_1m_2} > 0,$$  

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\[
(j_{31} - j_{42})^2 + 4j_{32}j_{41} > 0,
\]

(27)

More,
\[
|j_{31} + j_{42}| > \sqrt{(j_{31} - j_{42})^2 + 4j_{32}j_{41}}
\]

(28)

because it is equivalent to
\[
j_{31}j_{42} > j_{32}j_{41},
\]

(29)

that is
\[
\frac{k_1 k_2}{m_1 m_2} + \frac{k^2}{m_1 m_2} + \frac{2v}{\bar{\xi}^3} \frac{k}{m_2} > \frac{k^2}{m_1 m_2}.
\]

(30)

The previous relations assure us that \( \lambda_1^2 < 0, \lambda_2^2 < 0 \) so that the roots of the characteristic equation (23) are all pure imaginary. The equilibrium position \( (\bar{\xi}_1, \bar{\xi}_2, 0, 0) \) is simply stable.

5. The stability of the motion

Let \((\xi_1, \xi_2, \xi_3, \xi_4)\) a solution of the system (5) and \((u_1, u_2, u_3, u_4)\) a deviation sufficiently small in its norm. We can write
\[
\dot{\xi}_1 + \dot{u}_1 = \xi_3 + u_3, \quad \dot{\xi}_2 + \dot{u}_2 = \xi_4 + u_4,
\]
\[
\dot{\xi}_3 + \dot{u}_3 = \frac{1}{m_1} \left[ -k_1 (\xi_1 + u_1) + \frac{\varepsilon_1}{(\xi_1 + u_1)^2} + k(\xi_2 - \xi_1 + u_2 - u_1) + m_1 g \right],
\]
\[
\dot{\xi}_4 + \dot{u}_4 = \frac{1}{m_2} \left[ m_2 g - k(\xi_2 - \xi_1 + u_2 - u_1) \right].
\]

(31)

Since \( u_1 \ll \xi_1 \) we can approximate
\[
\frac{\varepsilon_1}{(\xi_1 + u_1)^2} \approx \frac{\varepsilon_1}{\xi_1^2} - \frac{2\varepsilon_1}{\xi_1^3} u_1.
\]

(32)

Keeping into account that \((\xi_1, \xi_2, \xi_3, \xi_4)\) is a solution of the system (5), from the relations (31) and (32) we obtain the system in deviations
\[
\dot{u}_1 = u_3, \quad \dot{u}_2 = u_4, \quad \dot{u}_3 = \frac{1}{m_1} \left[ -k_1 u_1 - \frac{2\varepsilon_1 u_1}{\xi_1^3} + k(u_2 - u_1) \right], \quad \dot{u}_4 = \frac{1}{m_2} \left[ -k(u_2 - u_1) \right],
\]

(33)

wherefrom
\[
m_1 \ddot{u}_1 = -k_1 u_1 - \frac{2\varepsilon_1 u_1}{\xi_1^3} + k(u_2 - u_1), \quad m_2 \ddot{u}_2 = -ku_2 + ku_1.
\]

(34)

From the second relation (34) we find
\[ u_1 = m_2 \dot{u}_2 + u_2 \, , \quad \ddot{u}_1 = \frac{m_2 u_2^{(iv)}}{k} + \ddot{u}_2 \, , \]  

the first relation (34) offering now  

\[ \frac{m_1 m_2}{k} u_2^{(iv)} + \left[ \frac{m_2}{k} \left( k_1 + k + \frac{2\epsilon_1}{\xi_1^3} \right) + m_1 \right] \ddot{u}_2 + ku_2 = 0 \, . \]  

The characteristic equation reads now  

\[ \frac{m_1 m_2}{k} r^4 + \left[ \frac{m_2}{k} \left( k_1 + k + \frac{2\epsilon_1}{\xi_1^3} \right) + m_1 \right] r^2 + k = 0 \, . \]  

The discriminate of this equation is  

\[ \Delta = \left[ \frac{m_2}{k} \left( k_1 + k + \frac{2\epsilon_1}{\xi_1^3} \right) + m_1 \right]^2 - 4m_1 m_2 \]  

\[ = (m_2 - m_1)^2 + \left[ \frac{m_2}{k} \left( k_1 + k \right) ^2 + 2(m_2 + m_1) \frac{m_2}{k} \left( k_1 + \frac{2\epsilon_1}{\xi_1^3} \right) \right] > 0 \]  

and, in addition,  

\[ \Delta < \left[ \frac{m_2}{k} \left( k_1 + k + \frac{2\epsilon_1}{\xi_1^3} \right) + m_1 \right]^2 \, . \]  

Keeping into account that  

\[ a = \frac{m_2}{k} \left( k_1 + k + \frac{2\epsilon_1}{\xi_1^3} \right) + m_1 > 0 \]  

it immediately results that \( r_1^2 < 0 \), \( r_2^2 < 0 \) so that the roots of the characteristic equation (37) are all pure imaginary, the motion being stable, but not asymptotically stable. The solution of the equation (36) is  

\[ u_2 = C_1 \cos \left( \frac{a - \sqrt{\Delta}}{2k} t + \varphi_1 \right) + C_2 \cos \left( \frac{a + \sqrt{\Delta}}{2k} t + \varphi_2 \right) \, . \]  

By twice derivation of the expression (41), we obtain  

\[ \ddot{u}_2 = \frac{a - \sqrt{\Delta}}{2k} C_1 \cos \left( \frac{a - \sqrt{\Delta}}{2k} t + \varphi_1 \right) - \frac{a + \sqrt{\Delta}}{2k} C_2 \cos \left( \frac{a + \sqrt{\Delta}}{2k} t + \varphi_2 \right) \]  

and from the first relation (35) it results
\[
u_i = \left( -\frac{m_2}{k} \frac{a - \sqrt{\Delta}}{2k} + 1 \right) C_1 \cos \left( \frac{a - \sqrt{\Delta}}{2k} t + \varphi_1 \right) + \left( -\frac{m_2}{k} \frac{a + \sqrt{\Delta}}{2k} + 1 \right) C_2 \cos \left( \frac{a + \sqrt{\Delta}}{2k} t + \varphi_2 \right).
\] (43)

Everywhere \( C_1, C_2, \varphi_1 \) and \( \varphi_2 \) are constants of integration, which result from the initial conditions \( u_1(0) = u_{i0}, \ u_2(0) = u_{20}, \ \dot{u}_1(0) = \ddot{u}_{i0}, \ \dot{u}_2(0) = \ddot{u}_{20} \).

The expressions (41) and (43) approximate the solution of the system in deviations (31).

6. The small oscillations around the equilibrium position

These can be obtained as a particular case of the previous paragraph for
\[
\xi_i = \bar{\xi}_i.
\] (44)

Result the eigenpulsations
\[
\omega_1 = \sqrt{\frac{m_2}{k} \left( k_1 + k + \frac{2\varepsilon_1}{\bar{\xi}_1^3} \right) + m_1 - \sqrt{\left[ \frac{m_2}{k} \left( k_1 + k + \frac{2\varepsilon_1}{\bar{\xi}_1^3} \right) + m_1 \right]^2 - 4m_1m_2} \frac{2k}{m_1m_2},
\] (45)
\[
\omega_2 = \sqrt{\frac{m_2}{k} \left( k_1 + k + \frac{2\varepsilon_1}{\bar{\xi}_1^3} \right) + m_1 + \sqrt{\left[ \frac{m_2}{k} \left( k_1 + k + \frac{2\varepsilon_1}{\bar{\xi}_1^3} \right) + m_1 \right]^2 - 4m_1m_2} \frac{2k}{m_1m_2}.
\]

7. Comparison with the linear case

The linear case is obtained for \( \varepsilon_1 = 0 \).

The equation (7) writes
\[
-k_1\bar{\xi}_1 + (m_1 + m_2)g = 0,
\] (46)

with the solution
\[
\bar{\xi}_1^{(i)} = \frac{m_1 + m_2}{k_1} g.
\] (47)

One observes that \( \bar{\xi}_1^{(i)} < \bar{\xi}_1 \) for which holds true the relation (15).

The relation (10) offers
\[
\bar{\xi}_2^{(i)} = \frac{m_2g}{k} + \frac{m_1 + m_2}{k_1} g
\] (48)
and therefore $\xi_2^{(i)} < \xi_2$, too.

The equilibrium remains again simply stable because the relations (25)-(30) still hold true.

The motion is again simply stable and we have in addition

$$\Delta^{(i)} = \left( m_2 + m_1 + \frac{m_2 k_1}{k} \right)^2 - 4m_1 m_2$$

$$= (m_2 - m_1)^2 + \left( \frac{m_2 k_1}{k} \right)^2 + 2(m_2 + m_1) m_1 \frac{m_2 k_1}{k} ,$$

$$a^{(i)} = m_2 + m_1 + \frac{m_2 k_1}{k} ,$$

with

$$\Delta^{(i)} > 0 , \quad \Delta^{(i)} < \Delta , \quad a^{(i)} > 0 , \quad a^{(i)} < a .$$

The eigenpulsations are

$$\omega_1^{(i)} = \sqrt{\frac{m_2 + m_1 + \frac{m_2 k_1}{k} - \sqrt{\left( m_2 + m_1 + \frac{m_2 k_1}{k} \right)^2 - 4m_1 m_2}}{2k \over m_1 m_2}} ,$$

$$\omega_2^{(i)} = \sqrt{\frac{m_2 + m_1 + \frac{m_2 k_1}{k} + \sqrt{\left( m_2 + m_1 + \frac{m_2 k_1}{k} \right)^2 - 4m_1 m_2}}{2k \over m_1 m_2}} .$$

8. Numerical application

Let us consider the case

$$m_1 = 2000 \text{ kg} , \quad m_2 = 1000 \text{ kg} , \quad g = 10 \text{ m/s}^2 , \quad k_1 = 10^6 \text{ N/m} , \quad \varepsilon_1 = 700 \text{ Nm}^2 , \quad k = 10^5 \text{ N/m} .$$

The equation (8) leads us to

$$10^6 \xi_1^3 - 3000 \times 10 \xi_1^2 - 700 = 0$$

with the solution

$$\xi_1 = 0.1 \text{ m} .$$

The relation (10) offers us

$$\xi_2 = \frac{1000 \times 10}{10^5} + 0.1 = 0.2 \text{ m} .$$

The expression (15) assures us that
\[ 0.1 \text{ m} > \frac{2 \times (2000 + 1000) \times 10}{3 \times 10^6} = 0.02 \text{ m}. \] 

The equations (17)-(20) lead us to

\[ j_{31} = \frac{1}{2000} \times \left( -10^6 - 10^5 - \frac{2 \times 700}{0.1^3} \right) = -1250, \quad j_{32} = \frac{10^5}{2000} = 50, \quad j_{41} = \frac{10^5}{1000} = 100, \quad j_{42} = -\frac{10^5}{2000} = -50, \]

the roots of the characteristic equation (23) being (24)

\[ \lambda_1^2 = -1250 - 50 \pm \sqrt{(-1250 + 50)^2 + 4 \times 50 \times 100} \quad \lambda_2^2 = -45.848, \quad \lambda_3^2 = -1254.152. \]

The parameters \( a \) and \( \Delta \) are

\[ a = \frac{1000}{10^5} \left( 10^6 + 10^5 + \frac{2 \times 700}{0.1^3} \right) + 2000 = 27000, \]
\[ \Delta = \left[ \frac{1000}{10^5} \left( 10^6 + 10^5 + \frac{2 \times 700}{0.1^3} \right) + 2000 \right]^2 - 4 \times 2000 \times 1000 = 721000000. \]

The eigenpulsations read

\[ \omega_1 = \sqrt{\frac{27000 - \sqrt{721000000}}{2 \times 10^5}} = 38.543 \text{ s}^{-1}, \quad \omega_2 = \sqrt{\frac{27000 + \sqrt{721000000}}{2 \times 10^5}} = 733.835 \text{ s}^{-1}. \]

In the linear case we have

\[ \xi_{11}^{(i)} = \frac{2000 + 1000}{10^6} \times 10 = 0.03 \text{ m}, \quad \xi_{22}^{(i)} = \frac{1000 \times 10}{10^5} + \frac{2000 + 1000}{10^6} \times 10 = 0.13 \text{ m}, \]
\[ a^{(i)} = 1000 + 2000 + \frac{1000 \times 10^6}{10^5} = 13000, \]
\[ \Delta^{(i)} = \left( 1000 + 2000 + \frac{1000 \times 10^6}{10^5} \right)^2 - 4 \times 2000 \times 1000 = 161000000, \]
\[ \omega_1^{(i)} = \sqrt{\frac{13000 - \sqrt{161000000}}{2 \times 10^8}} = 55.805 \text{ s}^{-1}, \quad \omega_2^{(i)} = \sqrt{\frac{13000 + \sqrt{161000000}}{2 \times 10^8}} = 506.839 \text{ s}^{-1}. \]

One observes that

\[ \omega_1 < \omega_1^{(i)}, \quad \omega_2 > \omega_2^{(i)}, \]

so that the non-linearity has as effect the increasing of the domain of pulsations where the resonance doesn’t appear.
Figure 3 – The time history for the system with parameters (53) and the initial conditions (68).

Figure 4 – The time history for the system with parameters (53) and the initial conditions (69).

In figure 3 are represented the diagrams $\xi_i(t), i = 1, 4$ for the parameters (53) the initial conditions being

$$\xi_{10} = 0.11 \text{ m}, \quad \xi_{20} = 0.19 \text{ m}, \quad \xi_{30} = 0, \quad \xi_{40} = 0.$$

(68)
Figure 5 – The time history for the system with parameters (53) and the initial conditions (70).

Figure 6 – The deviated motion between the systems captured in figures 4 and 5.

In figure 4 are represented the same diagrams for

\[ \xi_{10} = 0.15 \text{ m}, \quad \xi_{20} = 0.15 \text{ m}, \quad \xi_{30} = 0.1 \text{ m/s}, \quad \xi_{40} = 0, \]

(69)
in figure 5 for
\[ \xi_{10} = 0.151 \text{ m}, \quad \xi_{20} = 0.149 \text{ m}, \quad \xi_{30} = 0.1 \text{ m/s}, \quad \xi_{40} = 0.02 \text{ m/s}, \quad (70) \]
and in the figure 6 the deviated motion between the two cases from figures 5 and 6.

9. Conclusions

In this paper we presented a study concerning the influence of the non-linear neo-Hookean elements on the stability of the system machine-tool-foundation. We proved that both the equilibrium and the motion are simply stable and the neo-Hookean element increases the safety domain where the resonance doesn’t appear.

References


