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On the Use of Boundary Integral Equations and Linear Operators in Room Acoustics

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RESUMO: Este artigo introduz alguns conceitos de base sobre equações integrais e mostra a sua aplicação na determinação da propagação de energia acústica em recintos fechados.
Mostra-se que as equações integrais resultantes podem ser formuladas na linguagem dos operadores lineares, daí resultando uma notação simplificada em que as propriedades algébricas das equações que determinam a propagação da energia são mais facilmente caracterizadas.
São apresentados alguns métodos gerais de resolução de equações integrais tal como métodos aproximados e métodos de bases vectoriais finitas.
Apresentam-se, ainda, as definições necessárias envolvidas na descrição de campos de energia acústica, que servem de ponto de partida à aplicação das técnicas apresentadas.

ABSTRACT: This paper introduces some basic concepts of boundary integral equations and their application for the determination of the propagation of sound energy inside enclosures.
Linear operators are shown to provide a simplified notation and to emphasize the algebraic properties of the resulting integral equations.
Some general methods of solving linear operator and integral equations are reviewed and discussed, such as approximation methods and finite basis methods.
In addition, some of the necessary definitions involved in describing acoustic energy fields for applying these techniques in the field of room acoustics prediction are presented.

1. INTRODUCTION

Energy based methods offer an interesting and valid alternative mid and high frequency technique to classical predictive methods such as FEM, BEM and others. In fact, when the acoustic superposition of waves can be thought as being accomplished by incoherent components, then energy methods can be applied in a diverse range of problems.
In this paper, one starts by defining a set of acoustic variables that are solely derived from energy quantities, and then an energy balance equation is set up, that portrays the complete behaviour of the acoustic energy propagation inside enclosures.
Boundary integral equations have been derived earlier in some fields of acoustics. One example is Kuttruff's Integral Equation [1, 2] that determines the propagation of energy inside rooms with walls that reflect the sound energy in a purely diffuse way. However, to the authors knowledge, no generalization has been made for the general cases of arbitrary reflection laws. In addition, some work has been done by applying the finite element method to Kuttruff's equation, in the form of the Radiosity Method, but without a formal analysis of the underlying assumptions.

Finally, some work has been directed towards the development and application of Finite Element and Monte Carlo Methods (in the form of ray or cone tracing methods, for example), but the continuous infinite-dimensional case has not been addressed completely.
It is the objective of this paper to give some insight of the general continuous problem, resorting to concepts of functional analysis, and also to present some general methods to solve the problem.
All the quantities mentioned in this paper are dependent on the frequency of the acoustic disturbance.

2. THEORETICAL FRAMEWORK

2.1. Definitions:

- ***Angular Acoustic Intensity*** *(or Acoustic Radiance)*

$$I_\Omega(\underset{\sim}{\mathbf{r}},\theta,\varphi) = I_\Omega(\underset{\sim}{\mathbf{r}},\Omega) \tag{1}$$

is the amount of acoustic energy at some point ***r*** into a specified direction given by $\Omega = (\theta,\varphi)$ (related to a local coordinate system with the polar axis aligned with the surface's normal), per unit time, per unit area perpendicular to the direction of motion and per unit solid angle.

- ***Acoustic Power Flux*** into direction Ω

$$B_\Omega(\underset{\sim}{\mathbf{r}}) = I_\Omega(\underset{\sim}{\mathbf{r}},\Omega)\cos\theta d\Omega \tag{2}$$

- ***Acoustic Radiosity***

$$\begin{aligned} B(\underset{\sim}{\mathbf{r}}) &= \int_\Omega B_\Omega(\underset{\sim}{\mathbf{r}})d\Omega = \int_0^{2\pi}\int_0^{\pi/2} B_\Omega(\underset{\sim}{\mathbf{r}})\sin\theta\cos\theta d\theta d\varphi \\ &= \int_\Omega I_\Omega(\underset{\sim}{\mathbf{r}},\Omega)\cos\theta d\Omega = \int_0^{2\pi}\int_0^{\pi/2} I_\Omega(\underset{\sim}{\mathbf{r}},\Omega)\sin\theta\cos\theta d\theta d\varphi \end{aligned} \tag{3}$$

gives the total power leaving a point on a surface, per unit area, and is obtained by integrating over the upper hemisphere surrounding surface point ***r***.

- ***Surface Reflectivity***

A passive surface, which reflects the acoustic energy, such that when hit by an incident angular acoustic intensity, $I_\Omega(\underset{\sim}{\mathbf{r}},\theta_i,\varphi_i) = I_\Omega(\underset{\sim}{\mathbf{r}},\Omega_i)$, returns a reflected angular intensity, $I_\Omega(\underset{\sim}{\mathbf{r}},\Omega_o)$, can be characterized by a Surface Reflectivity Function (SRF) $R(\underset{\sim}{\mathbf{r}},\theta_o,\varphi_o,\theta_i,\varphi_i) = R(\underset{\sim}{\mathbf{r}},\Omega_o,\Omega_i)$ determined by:

$$I_{\Omega}^{Total}(\underset{\sim}{\mathbf{r}},\Omega_o)=\int_{\Omega} R(\underset{\sim}{\mathbf{r}},\Omega_o,\Omega_i)I_{\Omega}(\underset{\sim}{\mathbf{r}},\Omega_i)\cos\theta_i d\Omega \tag{4}$$

This expression allows the definition for the SRF, as the ratio between the angular acoustic intensity in the outgoing direction (subscript *o*) and the acoustic power flux in the incident direction (subscript *i*). It is therefore a function of both incoming and outgoing directions, and the preceding definition gives the following equation for the SRF:

$$R(\underset{\sim}{\mathbf{r}},\Omega_o,\Omega_i)=\frac{I_{\Omega}(\underset{\sim}{\mathbf{r}},\Omega_o)}{I_{\Omega}(\underset{\sim}{\mathbf{r}},\Omega_i)\cos\theta_i d\Omega_i} \tag{5}$$

In the most usual cases with physical meaning, the SRF obeys the "principle of detailed balance" with $R(\underset{\sim}{\mathbf{r}},\Omega_o,\Omega_i)=R(\underset{\sim}{\mathbf{r}},\Omega_i,\Omega_o)$. The function R is the most general expression for the acoustic energy reflection characteristic of a passive surface. The SRF is not necessarily a number between 0 and 1. By its definition, it can have any value between 0 and ∞, its units being [sr^{-1}].

- ***Directional-hemispherical Surface Reflectivity***

$$R_H(\underset{\sim}{\mathbf{r}},\Omega_i)=\frac{B(\underset{\sim}{\mathbf{r}})}{B_{\Omega}^{i}(\underset{\sim}{\mathbf{r}})}=\frac{\int_{\Omega} I_{\Omega}(\underset{\sim}{\mathbf{r}},\Omega_o)\cos\theta_o d\Omega_o}{I_{\Omega}(\underset{\sim}{\mathbf{r}},\Omega_i)\cos\theta_i d\Omega_i}=\int_{\Omega} R(\underset{\sim}{\mathbf{r}},\Omega_o,\Omega_i)\cos\theta_o d\Omega_o \tag{6}$$

is a dimensionless quantity between 0 and 1, expressing the ratio between the acoustic radiosity and the incoming acoustic power flux at ***r*** from direction Ω_i.

2.2. Energy Balance Equations

The general case of the energy balance at some surface element is determined by the following boundary integral equation:

$$I_{\Omega}(\underset{\sim}{\mathbf{r}},\Omega_o)=R(\underset{\sim}{\mathbf{r}},\Omega_o,\Omega_i^D)I_{\Omega}^{D}(\underset{\sim}{\mathbf{r}},\Omega_i^D)\cos\theta_i^D d\Omega_i^D+\int_{\Omega} R(\underset{\sim}{\mathbf{r}},\Omega_o,\Omega_i)I_{\Omega}(\underset{\sim}{\mathbf{r}},\Omega_i)\cos\theta_i d\Omega \tag{7}$$

where $I_{\Omega}^{D}(\underset{\sim}{\mathbf{r}},\Omega_i^D)$ refers to the angular acoustic intensity arriving at some boundary point ***r*** from direction Ω_i^D due to the direct field of a sound source located inside the enclosure. The hemispherical integral in equation (7) can be rewritten in terms of a surface integral over the enclosure's boundary. If ***s*** is the point on the boundary that is visible from ***r*** in the direction Ω_i, then ***r*** is also visible from ***s*** in a complementary direction Ω_i^* and according to the dissipation of the angular acoustic intensity in the air, one has:

$$I_{\Omega}(\underset{\sim}{\mathbf{r}},\Omega_i) = I_{\Omega}(\underset{\sim}{\mathbf{s}},\Omega_i^*)e^{-m\|\underset{\sim}{\mathbf{r}}-\underset{\sim}{\mathbf{s}}\|} \tag{8}$$

where *m* equals the absorption coefficient in the air. Therefore, equation (8) is transformed into:

$$\begin{aligned} I_{\Omega}(\underset{\sim}{\mathbf{r}},\Omega_o) &= R(\underset{\sim}{\mathbf{r}},\Omega_o,\Omega_i^D)I_{\Omega}^D(\underset{\sim}{\mathbf{r}},\Omega_i^D)\cos\theta_i^D d\Omega_i^D + \\ &\iint_S R(\underset{\sim}{\mathbf{r}},\Omega_o,\Omega_i)I_{\Omega}(\underset{\sim}{\mathbf{s}},\Omega_i^*)e^{-m\|\underset{\sim}{\mathbf{r}}-\underset{\sim}{\mathbf{s}}\|}\frac{\cos\theta_i\cos\theta_i^*}{\left\|\underset{\sim}{\mathbf{r}}-\underset{\sim}{\mathbf{s}}\right\|^2}vis(\underset{\sim}{\mathbf{r}},\underset{\sim}{\mathbf{s}})d\underset{\sim}{\mathbf{s}} \end{aligned} \tag{9}$$

where vis(***r, s***) is a visibility function that returns 0 if both points are not visible to each other and 1 otherwise. Introducing the transport and reflection integral linear operator Ψ:

$$I_{\Omega}(\underset{\sim}{\mathbf{r}},\Omega_o) = R(\underset{\sim}{\mathbf{r}},\Omega_o,\Omega_i^D)I_{\Omega}^D(\underset{\sim}{\mathbf{r}},\Omega_i^D)\cos\theta_i^D d\Omega_i^D + (\Psi I_{\Omega})(\underset{\sim}{\mathbf{r}},\Omega_o) \tag{10}$$

or more compactly as an operator equation:

$$I_{\Omega} = I_{\Omega}^D + \Psi I_{\Omega} \tag{11}$$

2.3. Properties of the Operator Equation

Operator Ψ in equation (11) can be written as follows:

$$(\Psi I_{\Omega})(\underset{\sim}{\mathbf{r}},\Omega_o) = \iint_S K(\underset{\sim}{\mathbf{r}},\underset{\sim}{\mathbf{s}})I_{\Omega}(\underset{\sim}{\mathbf{s}},\Omega_i^*)d\underset{\sim}{\mathbf{s}} \tag{12}$$

where *K*(***r***, ***s***) is supposed to be a measurable function over the domain of integration and constitutes the kernel of operator Ψ. It can be easily shown, that for SRF's with physical meaning, kernel *K* belongs to the Hilbert Space L^2, and more precisely the following is true:

$$\iint_S \iint_S \left|K(\underset{\sim}{\mathbf{r}},\underset{\sim}{\mathbf{s}})\right|^2 d\underset{\sim}{\mathbf{r}}d\underset{\sim}{\mathbf{s}} < 1 \tag{13}$$

A kernel that belongs to space L^2 is denoted as a Hilbert-Schmidt kernel. Consequently, the norm of operator Ψ verifies [3]:

$$\|\Psi\| \le \sqrt{\iint_S \iint_S \left|K(\underset{\sim}{\mathbf{r}},\underset{\sim}{\mathbf{s}})\right|^2 d\underset{\sim}{\mathbf{r}}d\underset{\sim}{\mathbf{s}}} < 1 \tag{14}$$

and so the spectral radius of the transport and reflection operator is less than unity. In this conditions, there exists a bounded operator $(ID-\Psi)^{-1}$ [3], where ID is the identity operator, and one has:

$$(ID-\Psi)^{-1}=\sum_{j=0}^{\infty}\Psi^{j} \tag{15}$$

Therefore, the formal and exact solution of the Fredholm integral equation of the second kind [4] (11) is given by:

$$I_{\Omega}(\underset{\sim}{\mathbf{r}},\Omega_o)=((ID-\Psi)^{-1}I_{\Omega}^{D})(\underset{\sim}{\mathbf{r}},\Omega_o)=\sum_{j=0}^{\infty}(\Psi^{j}I_{\Omega}^{D})(\underset{\sim}{\mathbf{r}},\Omega_o) \tag{16}$$

what is normally known in the literature as a Neumann series.

2.4. Approximate solutions of equation (16)

2.4.1. Truncated Neumann Series and Iterated Kernels

Define $\Theta_n=\sum_{j=0}^{n}\Psi^{j}$ as the operator related to the truncated series after n terms. This operator will approximate the exact solution of equation (16). The error of the approximation is given by:

$$\left\|ID-\Psi-\Theta_n\right\|=\left\|\sum_{j=n+1}^{\infty}\Psi^{j}\right\|\le\sum_{j=n+1}^{\infty}\left\|\Psi^{j}\right\|\le\sum_{j=n+1}^{\infty}\left\|\Psi\right\|^{j}=\frac{\left\|\Psi\right\|^{n+1}}{1-\left\|\Psi\right\|} \tag{17}$$

and thus, the metric deviation of the approximate solution of equation (16) is:

$$\left\|I_{\Omega}(\underset{\sim}{\mathbf{r}},\Omega_o)-I_{\Omega}^{n}(\underset{\sim}{\mathbf{r}},\Omega_o)\right\|\le\frac{\left\|I_{\Omega}^{D}(\underset{\sim}{\mathbf{r}},\Omega_o)\right\|\left\|\Psi\right\|^{n+1}}{1-\left\|\Psi\right\|} \tag{18}$$

Therefore, the error associated with the truncated series is a function of the norm of the transport and reflection operator, which by its turn depends on the SRF. The less the magnitude of the SRF, the less the incurred error in the calculated series. The explicit representation of this solution can be written with the help of the iterated kernels of order n, corresponding to a particular Θ_n operator:

$$K_n(\underset{\sim}{\mathbf{r}},\underset{\sim}{\mathbf{u}})=\iint_{S}K(\underset{\sim}{\mathbf{r}},\underset{\sim}{\mathbf{s}})K_{n-1}(\underset{\sim}{\mathbf{s}},\underset{\sim}{\mathbf{u}})d\underset{\sim}{\mathbf{s}} \tag{19}$$

2.4.2. The Nyström Method

The Nyström Method, or Quadrature Method, is one of the most straightforward methods for solving integral equations. It exploits the similarity between the infinite-dimensional integral operator and the corresponding finite-dimensional matrix formulation. m points $\boldsymbol{p_j}$ are selected in the domain of operator $(ID-\Psi)^{-1}$ at which to approximate the solution of the acoustic angular intensity $\boldsymbol{I_\Omega}$:

$$\underset{\sim}{\mathbf{q}}_j = (ID-\Psi)^{-1}(\underset{\sim}{\mathbf{p}}_j) \quad : \qquad j = 1,2,...,m \tag{20}$$

where $\underset{\sim}{\mathbf{p}}_j = \{\underset{\sim}{\mathbf{r}}, \underset{\sim}{\boldsymbol{\omega}}\}$ is a vector in R^3 x S^2. The Nyström Method defines that the value of $\boldsymbol{I_\Omega}$ at $\boldsymbol{p}$ is approximated by a quadrature rule as:

$$I_\Omega(\underset{\sim}{\mathbf{p}}) \approx I_\Omega^D(\underset{\sim}{\mathbf{p}}) + \sum_{j=0}^{m} w_j(\underset{\sim}{\mathbf{p}}) K^{-1}(\underset{\sim}{\mathbf{p}}, \underset{\sim}{\mathbf{p}}_j)\underset{\sim}{\mathbf{q}}_j \tag{21}$$

where K^{-1} is the kernel associated with operator $(ID-\Psi)^{-1}$, and where the weights w_j define the quadrature rule. Writing this rule for all the selected m points $\boldsymbol{p_j}$

$$\underset{\sim}{\mathbf{q}}_j = I_\Omega^D(\underset{\sim}{\mathbf{p}}_j) + \sum_{i=0}^{m} w_i(\underset{\sim}{\mathbf{p}}_j) K^{-1}(\underset{\sim}{\mathbf{p}}_j, \underset{\sim}{\mathbf{p}}_i)\underset{\sim}{\mathbf{q}}_i \tag{22}$$

that can be cast into matrix form and solved by standard techniques.

2.4.3. Finite Basis Methods

The basic concept of the Finite Basis Methods is to approximate a function space with a finite-dimensional subspace, that will be the span of some finite collection of basis functions chosen for their convenient properties. The goal is to find n scalar values α_i so that:

$$I_\Omega{}^n(\underset{\sim}{\mathbf{p}}) = \sum_{i=1}^{n} \alpha_i b_i(\underset{\sim}{\mathbf{p}}) \tag{23}$$

is in some sense a good approximation of $\boldsymbol{I_\Omega}$ and where the b_i are the selected basis functions of the finite-dimensional subspace. For sake of brevity we introduce the operator Φ=ID-Ψ. Thus, the exact solution of equation (16) satisfies:

$$(\Phi I_\Omega)(\underset{\sim}{\mathbf{r}}, \Omega_o) = I_\Omega^D(\underset{\sim}{\mathbf{r}}, \Omega_o) \tag{24}$$

- **Point Collocation Method**

In this method, the approximate function $\boldsymbol{I_\Omega^n}$ is chosen from the n-dimensional subspace by requiring the transformed function Φ $\boldsymbol{I_\Omega^n}$ to attain the desired value at a finite number of collocation points. That is, one selects n points $\boldsymbol{p_j}$ from the domain and require that:

$$(\Phi I_{\Omega}{}^{n})(\underset{\sim}{\mathbf{p}}_j) = \hat{I}_{\Omega}{}^{n}(\underset{\sim}{\mathbf{p}}_j) = I_{\Omega}^{D}(\underset{\sim}{\mathbf{p}}_j) \qquad : \qquad j = 1, 2, ..., n \tag{25}$$

which results in a system of n linear equations in the coefficients α_i:

$$\sum_{i=1}^{n} \alpha_i (\Phi b_i)(\underset{\sim}{\mathbf{p}}_j) = I_{\Omega}^{D}(\underset{\sim}{\mathbf{p}}_j) \tag{26}$$

that can be rewritten in matrix form.
The Point Collocation Method does not enforce equality of $\boldsymbol{I_{\Omega}^{n}}$ and $\boldsymbol{I_{\Omega}}$ at the points $\boldsymbol{p_j}$ since it only constrains the values of the transformed function $\Phi\, \boldsymbol{I_{\Omega}^{n}}$. However, it can be shown, that for suitably chosen collocation points, the approximation will converge to the exact values as $n \to \infty$.
The usual Radiosity Method applied in Room Acoustics and Thermal Engineering is derived from the Point Collocation Method. This is achieved by restricting the surfaces to be pure diffuse reflectors and by subdividing the boundary into a collection of disjoint patches. Finally, the basis functions are defined to be piecewise constant over every single patch. In this greatly simplified case, the SRF becomes a multiplicative factor in front of the integrals of operator Φ, and the integrals become purely geometrical quantities (form factors), whereby system (26) can be easily solved for.

- **The Least Squares Method**

This method is an application of Hilbert Space methods to the solution of integral equations, where the goal is again to best approximate the exact solution. The criterion of best approximation is now determined by the least squares minimization. That is, one seeks an approximate function $\boldsymbol{I_{\Omega}^{n}}$ such that the Hilbert Space norm-2:

$$\left\| (\Phi I_{\Omega}{}^{n})(\underset{\sim}{\mathbf{p}}) - I_{\Omega}^{D}(\underset{\sim}{\mathbf{p}}) \right\|_2 \tag{27}$$

is minimized. It can be shown that this condition corresponds to the requirement that the residual $(\Phi I_{\Omega}{}^{n})(\underset{\sim}{\mathbf{p}}) - I_{\Omega}^{D}(\underset{\sim}{\mathbf{p}})$ be orthogonal to the subspace generated by the transformed basis functions. Therefore, one has:

$$\left\langle (\Phi I_{\Omega}{}^{n})(\underset{\sim}{\mathbf{p}}) - I_{\Omega}^{D}(\underset{\sim}{\mathbf{p}}) \,|\, (\Phi b_i)(\underset{\sim}{\mathbf{p}}) \right\rangle = 0 \tag{28}$$

which yields:

$$\left\langle I_{\Omega}^{D}(\underset{\sim}{\mathbf{p}}) \,|\, (\Phi b_j)(\underset{\sim}{\mathbf{p}}) \right\rangle = \sum_{i=1}^{n} \alpha_i \left\langle (\Phi b_i)(\underset{\sim}{\mathbf{p}}) \,|\, (\Phi b_j)(\underset{\sim}{\mathbf{p}}) \right\rangle \tag{29}$$

and that can again be written as a set of linear algebraic equations in the coefficients α_i. The matrix of the inner products is a Gram matrix, which is non-singular provided that the basis functions are linearly independent.

- **The Galerkin Method**

In the Galerkin Method the condition of best approximation corresponds to the requirement that the residual $(\Phi I_\Omega{}^n)(\underset{\sim}{\mathbf{p}}) - I_\Omega^D(\underset{\sim}{\mathbf{p}})$ be orthogonal to the subspace generated from the original basis functions. Thus, this method seeks a solution that is exact once operator Ψ and the direct angular intensity $\boldsymbol{I_\Omega^D}$ have had their ranges collapsed onto the chosen n-dimensional subspace. The Galerkin condition is therefore:

$$\left\langle (\Phi I_\Omega{}^n)(\underset{\sim}{\mathbf{p}}) - I_\Omega^D(\underset{\sim}{\mathbf{p}}) \,|\, b_i(\underset{\sim}{\mathbf{p}}) \right\rangle = 0 \tag{30}$$

which gives the following set of linear algebraic equations:

$$\left\langle I_\Omega^D(\underset{\sim}{\mathbf{p}}) \,|\, b_j(\underset{\sim}{\mathbf{p}}) \right\rangle = \sum_{i=1}^{n} \alpha_i \left\langle (\Phi b_i)(\underset{\sim}{\mathbf{p}}) \,|\, b_j(\underset{\sim}{\mathbf{p}}) \right\rangle \tag{31}$$

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