# THE SUITABILITY OF THE WHB METHOD TO FAST NUMERICAL ACOUSTIC PRESSURE CALCULATIONS

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## 1. INTRODUCTION

Most numerical treatments of sound propagation in deep water use ray methods whereas in shallow water one tends to favour 'exact' solutions to the wave equation although there is nearly always some limitation to the 'exactness'. As an alternative this paper investigates the suitability of the WKB method for numerical calculation of the coherent and incoherent mode sum in deep water where there are many modes. The method is well known[1,2] for bridging the conceptual gap between rays and modes, and it promises not only a wave treatment but a rapid numerical solution if one is prepared to accept small errors in intensity as frequency is lowered.

The incoherent mode sum was investigated in the high frequency limit in a companion paper[3]. The objectives here are to see the effects of a finite frequency on both the incoherent and coherent mode sum. It will be shown that there are many conditions under which frequency dependence is weak and the WKB amplitude (without phase) is a sufficient substitute for the normal modes. A demonstration of this will be given, for the special case of a parabolic duct, by comparing the mode sum derived from the exact Hermite polynomial solutions, the sum of the WKB modes, and an integral formulation based on the WKB amplitude.

Analytical results are extended numerically for the parabolic duct, but a number of general results are deduced.

## 2. FREQUENCY DEPENDENT FACTORS

The intensity I is based on the discrete sum of normal modes which, to take account of the coherent or incoherent sum, can be written more conveniently in the form of a local range average at $r_o$ over a distance L (actually an average of the quantity I×r).

$$I = \frac{2\pi}{r_o} \int \sum_n \sum_m \phi_n(z_s)\phi_n(z_r)\phi_m(z_s)\phi_m(z_r)e^{i(K_n-K_m)r} \frac{e^{-(r-r_o)^2/L^2}}{(K_n K_m)^{\frac{1}{2}} \sqrt{\pi} L} dr \qquad (1)$$

Integrating in r gives

$$I=\frac{2\pi}{r_o}\sum_n\sum_m \phi_n(z_s)\phi_n(z_r)\phi_m(z_s)\phi_m(z_r)(K_nK_m)^{-\frac{1}{2}}\, e^{i(K_n-K_m)r_o}\, e^{-(K_n-K_m)^2L^2/4} \quad (2)$$

The coherent, unaveraged sum is given by putting L=o which produces the usual double sum, and the incoherent sum is given by putting $L\to\infty$ which only retains the terms with n=m, i.e. a single sum[3].

It was shown[3] that in the high frequency limit the incoherent sum could be evaluated by using a convenient trick for evaluating the normalisation of each mode and then converting the sum to an integral in wavenumber K. As frequency is lowered several issues become important in determining the behaviour. These are:

- the validity of the WKB approximation itself
- the accuracy of each mode's wavenumber (or phase velocity)
- the accuracy of each mode's normalisation
- the number of remaining modes
- the shape of the low order modes
- the accuracy of the integral in K

## 2.1 WKB Validity

The usual condition for validity of the WKB approximation is[4]

$$|(kdk/dz)/(k^2-K^2)^{3/2}|<<1 \quad (3)$$

where $k(z)=2\pi f/c(z)$ is the wavenumber profile and K is the mode's wavenumber or eigenvalue. Clearly WKB is invalid when K=k(z) (the 'edge' of the duct) and whenever k(z) is rapidly changing. This is not a problem at high frequency because the wavenumber (k(z)) up to which validity is assured approaches indefinitely close to K. When k(z) is close to K, $(k^2-K^2)$ will nearly always be more or less linear in z, and the solution is then an Airy function which can be fitted to the end of the WKB solution[4]. Indeed this match is essential for the derivation of the 'phase integral'. Figure 1(a) shows a sequence of modes (incidentally for a parabolic duct) and one can imagine the end cycle on each as an Airy function merging into the WKB solution towards the middle of the duct. Although a good fit in an arbitrary profile is feasible provided the mode has a number of cycles it is clearly difficult for the first mode.

## 2.2 WKB Phase Accuracy

It is well known[2] that the absolute phase of WKB modes deviates slightly from the phase of the exact modes so that beyond a certain range the WKB acoustic pressure is incorrect. However, it is the relative phases (the difference between one mode and adjacent ones that produce long range intensity structure such as convergence zones[5,6,7], and as long as they are reasonably accurate the intensity will not be affected.

# 3. MODE NORMALISATION

In the companion paper[3] it was shown that in the high frequency limit the WKB mode shape may be assumed up to the point where k(z)=K and that, for the purposes of integration, the $\cos^2$ oscillations could be ignored and their mean taken as ½. Thus the normalisation constant $N_n^2$ for the nth mode is

$$N_n^2 = \tfrac{1}{2}\int (k^2(z)-K_n^2)^{-\frac{1}{2}}\, dz \quad (4)$$

Independently of frequency the latter integral was shown to equal

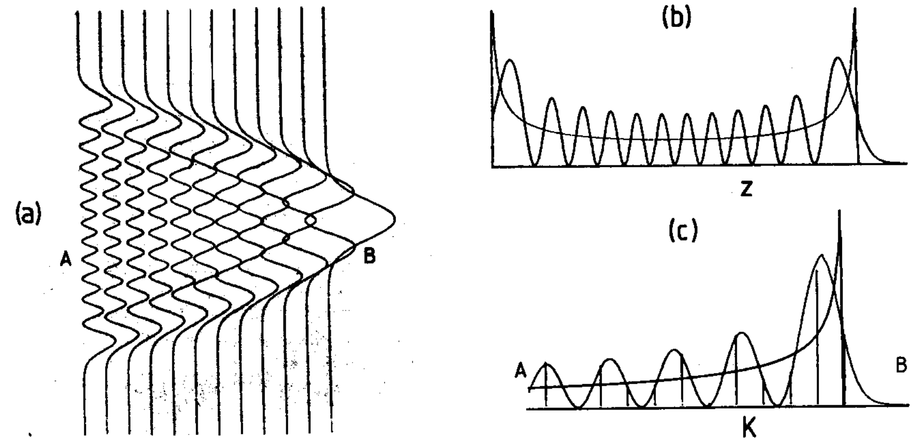


Fig. 1(a) Waterfall display of mode functions vs. z; (b) the exact normalisation integrand cf. WKB modulus integrand; (c) mode amplitude as a function of wavenumber K (along line A-B in Fig 1(a)).

$-\pi/(2K_n dK_n/dn)$ where dK/dn is the reciprocal of the derivative in K of the phase integral (which can also be viewed as the mode spacing in wavenumber). This equality is important for the conversion of a mode sum into an integral in K.

Although it appears unlikely it will be shown below that Eqn 4 deviates by only a small percentage from the true normalisation for the first 2 or 3 modes and is thereafter negligibly different independently of frequency despite the gross difference in curves being integrated (Fig 1(b). The demonstration is for a parabolic duct but the result is not significantly different for a general duct.

In a parabolic duct with $k^2(z)=k_1^2(1-a^2z^2)$ the exact normalised modes are given in terms of Hermite polynomials[8,7] as

$$\phi_n^H(z)=H_n(\beta z)e^{-(\beta z)^2/2}\,\beta^{\frac{1}{2}}\,(2^n n!\sqrt{\pi})^{-\frac{1}{2}} \tag{5}$$

where $\beta^2=ak_1$ and the frequency dependence is implicit in $k_1=2\pi f/c_1$.

A comparable WKB solution can be set up by matching the amplitude at the centre of the duct (z=0) for symmetric modes (even n, in this convention). Thus using the relation[8] $H_n(0)=2^{n/2}(n-1)!!$

$$\phi_n^W(z) = A\cos\left(\int(k^2-K^2)^{\frac{1}{2}}dz\right)/(k^2-K^2)^{1/4} \tag{6}$$

with $A=(k_1^2-K^2)^{1/4}(n-1)!!\ \ (\beta/n!\pi^{\frac{1}{2}})^{\frac{1}{2}}$

Performing the pseudo-normalisation integral for the WKB modulus function as in Eqn 4 now gives

$$I=\tfrac{1}{2}\int(A^2/(k^2-K^2)^{\frac{1}{2}})dz$$

$$= (\pi(2n+1))^{\frac{1}{2}}2^{-n-1}\ n!/(n/2)!(n/2)! \tag{7}$$

For n=2, 4, 6 the deviations from 1 are .009, .003 and .0015 respectively, and using the second order Stirling approximation Eqn 7 is for large n

$$I = 1-1/16(n^2+n/3) \tag{8}$$

The first surprising point is that there is no frequency dependence at all, and the second is that even for the third mode (n=2) the integral differs by less than 1% from the true normalisation.

Although the comparison has been between the true normalisation and the integral of the WKB modulus function it is independently known that the complete WKB solution is a good approximation to the exact solution in the middle region of the duct up to about the zero before the last peak. Therefore the discrepancy (Eqn 7 or 8) must be mainly due to the more or less linear region of $k^2(z)$ at each edge of the duct and would be unaffected by deepening or modifying $k(z)$ in the middle of the duct. At worst for a different shaped duct one might expect changes in detail to Eqn 7 but not changes in order of magnitude of the difference. The derivation is therefore valid for nearly any profile, and bearing in mind that the first few modes will be swamped by higher order ones the normalisation in Eqn 4 can be used at all frequencies and for all mode numbers, and one can still exchange the mode sum for an integral over wavenumber with negligible error.

## 4. INCOHERENT INTENSITY COMPARISON

A number of forms are available for the mode function: the true one (in this case Hermite polynomials but not usually available), the WKB solution combined with an Airy function (or even Hermite polynomial) tail and the WKB modulus or amplitude function. These may be summed or integrated in wavenumber (assuming a mode number continuum)[3], and the way the mode amplitude varies with mode number or wavenumber at a fixed depth is shown in Fig 1(c) and can be seen by following the line A-B in Fig 1(a). The clear computational advantage of integration for the incoherent sum is that the variation of mode amplitude with K is all that is required and the modal eigenvalues are not needed. Similarly the coherent sum needs only eigenvalue differences.

Now various sums and integrals can be compared with the exact solution to see how serious the differences are particularly with the crude, easily calculated ones.

### 4.1 Incoherent Intensity Far From a Caustic

Figure 2(a) shows incoherent intensity versus frequency away from a caustic, i.e $k(z_r) \neq k(z_s)$ (source depth $z_s$ = 200m and receiver depth $z_r$ = 100m) in a parabolic duct, truncated in wavenumber at $k_o$. The solid line is the exact solution, an incoherent sum (Eqn 2 with $L \to \infty$) with Hermite polynomials (Eqn 5). The horizontal line is drawn at the value of the high frequency analytic solution[3]

$$I = \frac{a}{\pi r} \ln \frac{2[(k_s^2 - k_o^2)(k_r^2 - k_o^2)]^{\frac{1}{2}} + (k_s^2 - k_o^2) + (k_r^2 - k_o^2)}{|k_r^2 - k_s^2|} \qquad (9)$$

although it is, of course, only strictly valid for $f \to \infty$.

Nevertheless, the striking point is that, in practice, agreement is very good down to frequencies fairly near the cut-off. In other words, a large number of modes is not essential for Eqn 9 after all, and errors are less than 1dB with just 2 or 3 modes. The reason for this is that the integral of the smooth modulus function in K is close to the integral of the oscillatory function in K in the same way as was found with the normalisation integrals in z. Also, for a duct of general shape there is a simple general procedure[3] for calculating an equivalent high frequency formula to Eqn 9.

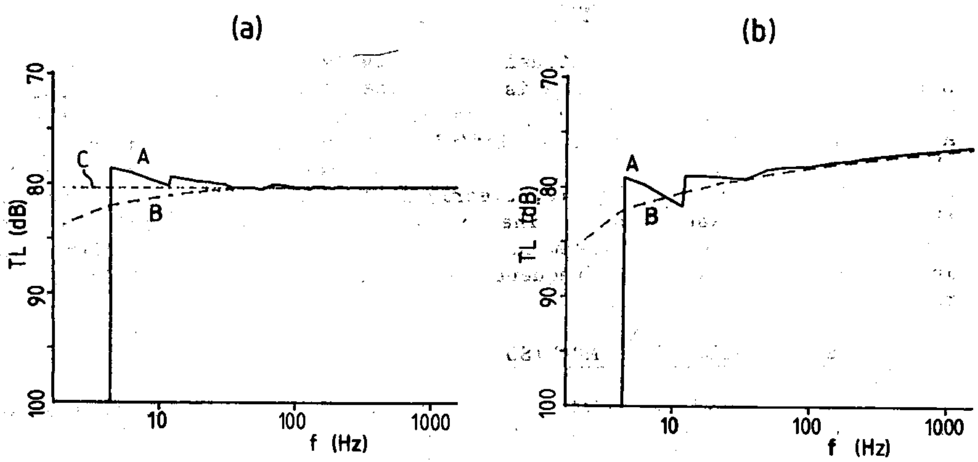


Fig. 2 Incoherent intensity for source at $z_s$=200m (a) away from the conjugate depth ($z_r$=100m) and (b) at the conjugate depth ($z_r$=200m) at range 27km. The exact sum, the WKB numerical integral and the WKB analytical (h.f.) formula are shown as A, B, C. The mode cut-off for a=2.3 $10^{-4}$ is 4.2 Hz.

An intermediate approach involving more computational effort but providing some frequency dependence is to perform the integral in K but use a general mode function composed of a matched WKB solution for the oscillatory part and an Airy function for the decaying part. Thus the integral becomes

$$I = \frac{-8}{\pi r_o}\int_{k_o}^{k_s-k_c} K(dK/dn)\cos^2(w_s-\pi/4)\cos^2(w_r-\pi/4)(k_r{}^2-K^2)^{-\frac{1}{2}}(k_s{}^2-K^2)^{-\frac{1}{2}}\,dK$$

$$+\int_{k_s-k_c}^{k_r-k_c} K(dK/dn)\cos^2(w_r-\pi/4)\ (k_r{}^2-K^2)^{-\frac{1}{2}}\ a_s\ Ai^2(y_s)dK$$

$$+\int_{k_r-k_c}^{\infty} K(dK/dn)\ \ a_s a_r\ Ai^2(y_s)\ Ai^2(y_r)dK \qquad (10)$$

where $w = \int_{z_B}^{z} (k(z)^2-K^2)^{\frac{1}{2}}dz$

$y = -(k^2-K^2)/(2kk')^{1/3}$

$a = \pi/(2kk')^{1/3}$

$k'$ is the gradient at z, and $k_c$ is the small wavenumber shift of the changeover below either $k_s$ or $k_r$ (taken to be about one quarter cycle (Fig 1(c)).

At high frequency (predominantly high mode number) the second and third integrals are negligible and $k_c \to 0$ resulting in the analytic solution Eqn 9. At lower frequencies numerical solution is shown in Fig 2(a) by the dashed line. In the first and second integrals the $\cos^2$ terms were again taken as ½ on average. The discrepancy at low frequency is mainly due to the difference between mode sum and mode integral and not the Airy/WKB matching.

## 4.2 Incoherent Intensity Near a Caustic

Since the incoherent intensity is equivalent to a range average with $L\to\infty$ it can be shown[9] that there is always the residual effect of a caustic when the receiver is at the conjugate depth of the source ($k(z_r)=k(z_s)$). At high frequency the resulting infinity in the intensity is obvious in the analytical solution, Eqn 9. However, at finite frequency the numerical integral form, Eqn 10, clearly is finite (since $k_c>0$) as must be the sum of the exact modes. The exact and the WKB/Airy[c] solutions are compared in Fig 2(b). Agreement is very good, and it is clear that intensity does rise with frequency although it is a surprisingly slow rise.

## 5. COHERENT INTENSITY COMPARISON

Although, at first sight, it looks as though the coherent mode sum can be converted into an integral in the same way as the incoherent sum, but the reason this is not so is that the mode spacings and their discreteness are precisely what generates structure in range. For instance, the spacing of adjacent modes is responsible for the convergence zones[5-7]. The only way the sum can be written in integral form is by retaining the discreteness in a separate function f(K) which is composed of a row of delta functions, one at the wavenumber eigenvalue of each mode. At high frequency this is

$$I = \frac{2}{\pi r_o}\left[\int_{k_o}^{k_s} g(K)f(K)e^{iKr}dK\right]^2 \qquad (11)$$

where $g(k) = \dfrac{K^{\frac{1}{2}}\cos(w_s-\pi/4)\cos(w_r-\pi/4)}{(k_s{}^2-K^2)^{1/4}(k_r{}^2-K^2)^{1/4}}$

This is messy to solve directly, but it does demonstrate that viewing the integral as the Fourier transform of the product of two functions f and g the coherent intensity is the convolution of the transforms of f and g, i.e.

$$I(r_o) \propto \int \bar{f}\,(r-r_o)\bar{g}(r)dr \qquad (12)$$

In other words, if f is a regular sequence of delta functions (as it is in a parabolic duct) with spacing $\Delta K$ in wavenumber then by using the Poisson sum formula $\bar{f}$ is a regular sequence of spikes in range with spacing $2\pi/\Delta K$. Since this is convolved with the function $\bar{g}(r)$ the intensity must be a function that repeats in range for ever with a repeat interval $2\pi/\Delta K$ (which can actually be shown to be the convergence distance for a parabolic duct).

The simple but computationally tedious method of calculating the coherent intensity is to insert joint WKB/Airy solutions into the coherent sum formula. This can be compared with the exact solution using Hermite polynomials, and the results are shown for source and receiver at different depths (100, 200m from the axis) in Fig 3(a) and the same depths (200m) in Fig 3(b), both at the first convergence range (27km since $a=2.3\ 10^{-4}$). Again deviations are less than 1dB from an octave above the mode cut-off upwards. As one expects, the intensity at a caustic or focus rises with frequency whereas it falls elsewhere. In contrast the frequency dependence of the incoherent intensity is very weak. At low frequencies (within a few octaves of cut-off) the convergence zones are not very pronounced and the analytic incoherent mode formula Eqn 9 is still surprisingly good for <u>all</u> cases, coherent or incoherent.

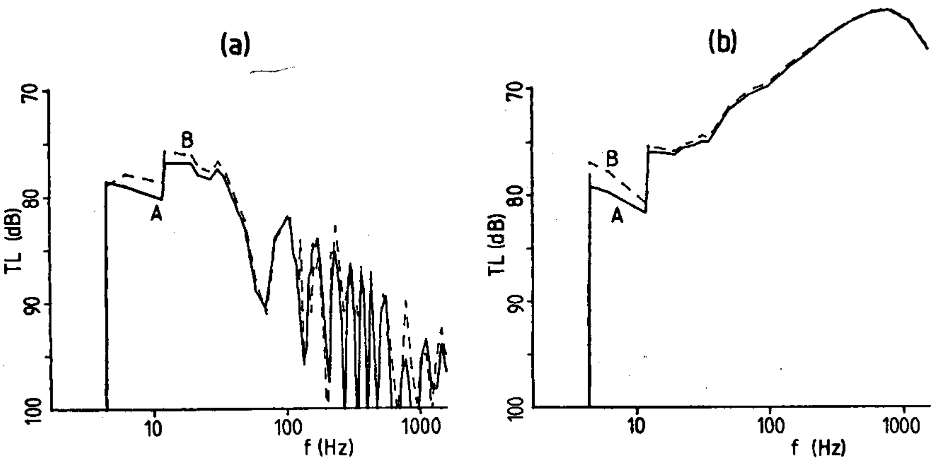


Fig. 3 Coherent intensity for source at $z_s$=200m (a) away from a caustic ($z_r$=100m, $r_o$=27km) and (b) at a caustic ($z_r$=200m, $r_o$=27km). The exact sum and the WKB sum using simplified normalisation are shown as A, B. Mode cut-off is 4.2Hz.

A more efficient approach is a hybrid sum and integral. Returning to Eqn 2 the double sum is separated into a sum in n and a sum in m-n. In this way all the mode phase differences are dealt with in one true sum. The remaining sum is then well behaved and can be converted to an integral which can be performed analytically or numerically.

At a high frequency the Airy function tails contribute a negligible amount and only the WKB cos terms are significant so that

$$I = \frac{8}{\pi r_o} \sum_{n=o}^{N} \frac{K_n \Delta K_n}{(k_s^2-K_n^2)^{\frac{1}{2}}(k_r^2-K_n^2)^{\frac{1}{2}}} J_n \tag{13}$$

where $J_n$ represents the sum in m-n (=j) given by

$$J_n=\Delta K_n \sum_{j=-n}^{N-n} \cos(s,n)\cos(s,m)\cos(r,n)\cos(r,m)e^{-i\Delta K_n r_o j}\, e^{-\Delta K_n^2 L^2 j^2/4} \tag{14}$$

and $\cos(s,n)=\cos\left(\int_{z_B}^{z_s} (k_s^2-K_n^2)dz-\pi/4\right)$ etc.

$\Delta K_n$ = dK/dn = mode spacing

Assuming that there is some averaging (L finite but possibly small) the limits of m-n (or j) will always be set by the gaussian in j and so the upper limits of the sum can be replaced by $\pm\infty$. The four cos terms can be taken in pairs and expressed in terms of j by using the identity cos A cos B = ½ (cos(A-B)+cos(A+B)) to give (in shorthand)

$$\cos(s,n)\ \cos(s,m)=\tfrac{1}{2}\ (\cos(G_s \Delta K_n j)+H_s) \tag{15}$$

where $$G_s=-K_n \int_{z_B}^{z_s} (k^2-K_n^2)^{-\frac{1}{2}}dz \tag{16}$$

$$H_s = \cos(2\int_{z_B}^{z_s}(k^2-K_n^2)^{\frac{1}{2}}dz-\pi/2) = \sin(2\int_{z_B}^{z_s}(k^2-K_n^2)^{\frac{1}{2}}dz \qquad (17)$$

Equation 14 can then be written as

$$J_n = \Delta K_n \sum_{j=-\infty}^{\infty} 1/4(\cos G_s \Delta K_n j + H_s)(\cos G_r \Delta K_n j + H_r) e^{-i\Delta K_n r_o j} e^{-\Delta K_n^2 L^2 j^2/4} \qquad (18)$$

and expanding the cos terms as exponentials each term can be written entirely in the form $\exp(iAj-Bj^2)$ which allows the sum to be solved (or converted) by the Poisson sum formula, i.e.

$$\sum_{-\infty}^{\infty} e^{iAj} e^{-Bj^2} = (\pi/B)^{\frac{1}{2}} \sum_{-\infty}^{\infty} e^{-(2m\pi+A)^2/(4B)^2}$$

The result is

$$J_n = (\pi^{\frac{1}{2}}/2L) \sum_{m=-\infty}^{\infty} [\ \sum_4 (1/4)\ e^{-(mr_c-r_o\pm G_s\pm G_r)^2/L^2}$$

$$+ \tfrac{1}{2}H_s \sum_2 e^{-(mr_c-r_o\pm G_r)^2/L^2} + \tfrac{1}{2}H_r \sum_2 e^{-(mr_c-r_o\pm G_s)^2/L^2}$$

$$+ H_s H_r\ e^{-(mr_c-r_o)^2/L^2}\ ] \qquad (19)$$

where the inner sum notation means to include terms with all possible signs in the exponent, and $r_c$ is the convergence distance $r_c=2\pi/\Delta K$. Although the outer sum is formally infinite the index m now represents convergence zone number so in practice only the range 0 to 2 or so is relevant.

Now that all oscillatory behaviour has been incorporated in Eqn 19, the original sum in n (Eqn 13) can be converted to an integral

$$I = \frac{8}{\pi r_o}\int_{k_o}^{k_s} K\ (k_s^2-K^2)^{-\frac{1}{2}}(k_r^2-K^2)^{-\frac{1}{2}} J dK \qquad (20)$$

In the high frequency limit with small L this (in combination with Eqn 19) reduces to the extremely simple ray intensity formula because $H_s$ and $H_r$, being oscillatory contribute nothing, and the remaining gaussians behave as delta functions in K (rather than range), one or more of which picks off a value of K (or effectively ray angle) in the integral. The condition that the exponent is zero is the same as the condition that there is an 'eigenray' between source and receiver. The condition for a caustic is that the exponent should remain zero for a finite range of K. In other words, the rate of change with K equals zero. This is exactly the ray condition for a caustic, and with L=0 this results in an infinite intensity as one would expect.

Note that although the range average length L has been considered as a provided input there are two other ways that the sum in Eqn 14 might be limited. One is through the limited number of modes N; thus an effective

L would be $L' \sim 4/N.\Delta K$. Another is through a slowly changing mode spacing. This would produce an extra term (~~by~~ further Taylor expanding $\Delta K_n$) of exp $(i\frac{1}{2}r_o d^2K/dn^2 j^2)$ which produces an effective length

$$L' = (2r_o\, d^2K/dn^2)^{\frac{1}{2}}/\Delta K = 2r_o\, [\int k^2(k^2-K^2)^{-3/2}dz/K\int(k^2-K^2)^{-\frac{1}{2}}dz]^{\frac{1}{2}}$$

A simple rule would be to use which ever is the bigger of the three in Eqn 19.

At lower frequencies the Airy function tails become more important, particularly near caustics, and so additional cross terms need to be considered. However, some range averaging will mean that large mode number differences can be ignored so that only two extra cross terms are important. Thus Eqn 2 is approximately

$$I = \frac{2\pi}{r_o}\int \Big[ \sum_{n=o}^{n_1}\sum_{m=o}^{n_1} D\, Ai(s,n)Ai(s,m)Ai(r,n)Ai(r,m)a_s a_r$$
$$+ \sum_{n=n_1}^{n_2}\sum_{m=n_1}^{n_2} D\, C(r,n)C(r,m)Ai(s,n)Ai(s,m)a_s$$
$$+ \sum_{n=n_2}^{N}\sum_{m=n_2}^{N} D\, C(s,n)C(s,m)C(r,n)C(r,m)\Big]\, dr \qquad (21)$$

where
$D=4K(dK/dn)^2\ e^{i(K_n-K_m)r_o}\ e^{-(r-r_o)^2/L^2}/\sqrt{\pi}L$, $C(s,n)=\cos(s,n)/(k_s{}^2-K_n{}^2)^{1/4}$ and the other terms are defined after Eqn 10. The mode numbers $n_1$, $n_2$ and N correspond to the wavenumbers $k_r-k_c$, $k_s-k_c$ and $k_o$ respectively.

The Airy function pairs are not as easily split into a constant and a mode-separation - dependent term as the cos terms were (in Eqn 15), but since their contribution only exists over a short range of mode numbers the terms can be assumed to be equally spaced in K. Thus the first double sum is identically the square of the discrete FT of Ai(s,K)Ai(r,K) which can be calculated by FFT with sample spacing $\Delta K$. The repeats at $(r_o-2\pi m/\Delta K)$ are the automatic consequence of aliasing. A similar approach is possible for the second double sum.

## 6. CONCLUSIONS

Some general formulations for the acoustic intensity have been derived from the coherent and incoherent mode sum, the objectives being to trade off computational complexity against accuracy. The mode sum can be expressed as an integral of a smoothly varying function that has the advantage that it does not require explicit knowledge of the modal eigenvalues. The investigation compared various constructions for the modes including WKB, WKB modulus and Airy function tails using mode sums and integrals with the exact solution for a parabolic duct. An arbitrary amount of range averaging can be included, if desired.

An interesting result is that the method used to calculate the mode normalisation integral for high mode number (i.e. using the WKB modulus function rather than WKB itself) is, in fact, accurate to less than 1% down to the third mode. This useful result and the relation between normalisation and mode spacing allows mode sums to be converted to integrals at virtually all frequencies.

The high frequency incoherent mode analytic formula is surprisingly accurate (1dB) (away from caustics) down to frequencies where there are only a couple of modes (an octave above cut-off). The WKB numerical integral is no better in this case but is clearly more generally applicable. Near a caustic the WKB/Airy integral agrees well with the exact solution and frequency dependence is surprisingly weak.

A straightforward WKB/Airy coherent mode sum (using the high frequency normalisation) compares well at low frequencies with the exact solution near or far from a caustic. Although the intensity rises with frequency near a caustic and falls elsewhere the simple analytical incoherent formula (Eqn 9) seems to be sufficient for all cases at low frequencies.

An alternative formulation sums the interfering mode contributions analytically leaving a numerical integral to be computed. This reduces exactly in the high frequency limit to the well known ray intensity formula. At lower frequencies it provides an economical method of calculating (range averaged or unaveraged) intensity since no reference is made to modal eigenvalues.

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