THE AVERAGE INTENSITY IN A DUCT USING THE WKB METHOD

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INTRODUCTION

In spite of the fact that there are many sophisticated propagation models available nowadays, such as PAREQ[1], IFD[2], FFP[3], and SAFARI[4], there are often occasions when a less detailed, more global understanding of what is going on is appropriate. In this paper some analytical solutions for averages of the acoustic intensity propagating in a duct are derived by using the WKB approximation for the normal mode shapes and inserting these in an incoherent mode sum. This approach is particularly useful in deep water where there is a large number of modes since the sum can be converted to an integral which is conducive to rapid computation and even analytical solution in some cases. Even so, an attempt is made to refrain from assuming any particular environment in the analysis until the very last opportunity.

Two types of average are considered here, one being taken over all source depths, the other over all source or receiver ranges. The source depth average gives some idea of the expected depth distribution of intensity at the receiver when the source depth is unknown, as is frequently the case in operational ASW scenarios. The range average is, in fact, an incoherent mode sum, and it is interesting that it gives identical formulae to those derived using an acoustic flux concept[5] derived from a ray treatment.

An advantage of the present technique is that the effects of finite frequencies can be included and also the solution is in a form that can easily be evaluated numerically. In a companion paper[6] the suitability of the WKB method is investigated numerically at finite frequencies and for the coherent pressure itself.

The validity of this approach depends partly on the WKB approximation which can only predict reasonably oscillatory modes and not the very low order ones. It also depends on the assumption of a large number of modes so that the mode sum can be converted to an integral (not to be confused with the continuous spectrum of modes). The method is therefore most suitable for deep water cases, and high frequencies if the simple first order WKB method is used. However, there is a better approximation[7] that allows frequency dependence to be included. Moreover, the accompanying paper[6] shows that this apparently necessary "large number of modes" is, in fact, quite small, and "high frequency" approaches such as WKB and rays can be used in some cases where there are only three of four modes in a duct.

Central to the method is a mathematical trick to facilitate normalisation of the modes which would otherwise be intractable analytically. This trick may also be applied to calculate the mode's contribution in a coherent sum for the pressure or unaveraged intensity.

2. NORMAL MODE FORMULAE

2.1 Incoherent Mode Sum (Range Average)

The usual discrete mode sum formula for the pressure p at range r and depth z_r relative to a standard pressure at one metre from the source at depth z_s in a stratified medium is

$$p = \pi i \sum_n \phi_n(z_s)\ \phi_n(z_r)\ H_o(K_n r) \tag{1}$$

where ϕ_n are the discrete mode amplitudes (assumed real throughout), K_n is the horizontal wave number and H_o is the Hankel function of the first kind of order zero. At ranges greater than a few wavelengths the Hankel function can be replaced by its asymptotic form, and

$$p = (2\pi i)^{\frac{1}{2}} \sum_n \phi_n(z_s)\phi_n(z_r)(K_n r)^{-\frac{1}{2}}\ e^{iK_n r} \tag{2}$$

In an incoherent addition the phase terms are assumed to be random so that Eqn 1 reduces to a formula for relative intensity I

$$I_r = \frac{2\pi}{r} \sum_n \phi_n^2(z_s)\phi_n^2(z_r)\ K_n^{-1} \tag{3}$$

Loss of coherence is equivalent to loss of structure in range, in particular, loss of convergence zones. The relation between ray convergence and normal mode relative phase has already been investigated by Nicholas and Uberall[8].

2.2 Depth Average

Equation 3 is the basis of the range average, and an equivalent formula can be found for the depth average. For this purpose it is necessary to assume that the source depth is uniformly distributed over all depths for which the modes have a significant amplitude. Depending on the velocity profile and the significance of the various mode contributions this may make the average rather unrealistic, but not necessarily so.

Returning to the coherent sum Eqn 2 the average intensity over depth H may be written as the integral of a double sum.

$$I_z = \frac{2\pi}{rH} \int_o^H \sum_m \sum_n \phi_m(z_s)\phi_m(z_r)\phi_n(z_s)\phi_n(z_r)(K_n K_m)^{-\frac{1}{2}}\ e^{i(K_m-K_n)r}\ dz_s \tag{4}$$

However, the orthognonality condition for all modes is that

$$\int \phi_m(z)\ \phi_n(z)\ dz = \delta(n-m)$$

where δ is the Kronecker delta, equal to 0 or 1. Therefore Eqn 4 becomes

$$I_z = \frac{2\pi}{rH} \sum_n \phi_n^2(z_r)/K_n \qquad (5)$$

This is now the basis of the depth average calculation.

3. WKB APPROXIMATION

Equations 3 and 5 can be evaluated by using the WKB approximation for the normal modes[9] ϕ_n. To make the mode normalisation tractable a trick is employed which results in a factor proportional to the rate of change of horizontal wave number K_n with mode number n. This is convenient because the mode sum can then easily be converted to a manageable integral over K_n.

The WKB solution for ϕ_n is

$$\phi_n(z) = (1/N_n h^{\frac{1}{2}}) \cos (w - \pi/4) \qquad (6)$$

where $h = (k^2(z) - K_n^2)^{\frac{1}{2}}$ and $w = \int^z h \, dz$ (7)

and $k(z) = 2\pi f/c(z)$ is the depth dependent wavenumber, and N_n is the, as yet, unknown normalisation.

The term w becomes the "phase integral" W

$$W = \int h \, dz \qquad (8)$$

if the integral is taken from the point where h=0 below the source and receiver to the point where h=0 above the source and receiver (assuming the source and receiver are in the same duct). It can be shown that [7,9] the phase integral is related to the mode number by

$$W = (n+\tfrac{1}{2})\pi \qquad (9)$$

The normalisation constant N_n is found by setting

$$\int \phi_n^2(z) \, dz = 1 = \int (1/N_n^2 h) \cos^2 (w-\pi/4) \, dz \qquad (10)$$

and for all but the low order modes the cosine term is highly oscillatory so that cosine squared can be replaced by its average value of one half. Thus

$$N_n^2 = \tfrac{1}{2} \int (k^2(z) - K_n^2)^{-\frac{1}{2}} \, dz \qquad (11)$$

Now, by coincidence this integral is related to the phase integral as follows. In the WKB approximation the mode amplitude goes to infinity at the depth where $k(z)=K_n$, (h=0). Outside these limits (where $k(z)<K_n$) the amplitude is zero. Therefore the depth limits in the phase integral and the normalisation integral are effectively the same, being the points where $k(z)=K_n$, say z_{Tn} and z_{Bn}.

Because the integrand is zero identically at z_{Tn} and z_{Bn} differentiating the phase integral with respect to K_n gives

$$\frac{dW}{dK_n} = -K_n \int_{z_{Bn}}^{z_{Tn}} (k^2(z)-K_n^2)^{-\frac{1}{2}} dz \qquad (12)$$

and so

$$\frac{dW}{dK_n} = -2K_n N_n^{\,2} \tag{13}$$

But from Eqn 9

$$\frac{dW}{dK_n} = \pi/(dK_n/dn) \tag{14}$$

where dK_n/dn is the wavenumber step between modes so that the normalisation is simplified to

$$N_n^{\,2} = -\pi/(2K_n\ dK_n/dn) \tag{15}$$

The normal modes may now be written as

$$\phi_n^{\,2}(z) = (-2/\pi)\ K_n(dK_n/dn)\ \cos^2\ (w-\pi/4)/(k^2(z)\ -\ K_n^{\,2})^{\frac{1}{2}} \tag{16}$$

or on average

$$\phi_n^{\,2}(z) = (-K_n dK_n/dn)/(\pi(k^2(z)-K_n^{\,2})^{\frac{1}{2}}) \tag{17}$$

4. UNIFORM SOURCE DEPTH DISTRIBUTION

Returning to the depth averaged formula Eqn 5 and inserting Eqn 17 produces

$$I_z = \frac{-2}{rH} \sum_n (dK_n/dn)\ (k^2(z_r)-K_n^{\,2})^{-\frac{1}{2}} \tag{18}$$

Now, if there is a large number of modes the mode number can be thought of as continuously variable, indeed, it has already been treated in this way in Eqn 14. In this case the summation can be treated as an integral by exchanging the Σ_n operator for $\int dn$. Thus (dropping the subscript n)

$$I_z = \frac{-2}{rH} \int_{n=o}^{n=N} (k^2(z_r)-K^2)^{-\frac{1}{2}}\ (dK/dn)\ dn \tag{19}$$

$$= \frac{2}{rH} \int_{K=k_{min}}^{k_{max}} (k^2(z_r)-K^2)^{-\frac{1}{2}}\ dK \tag{20}$$

The highest value that K can take is clearly $K=k(z_r)$ (corresponding to the point where the lowest possible order mode just gives a significant amplitude at the receiver depth). The lower limit of K is given by the value at the edge of the duct, say $k(z_0)$.

I_z can now be integrated, and

$$I_z = (2/rH)\ [\pi/2\ -\ \sin^{-1}\ (k(z_0)/k(z_r))] \tag{21}$$

which can be written more concisely as

$$I_z = (2/rH)\ \cos^{-1}\ (k(z_0)/k(z_r)) \tag{22}$$

This formula is true for all velocity profiles and only depends on the value of velocity at the receiver and at the edge of the duct.

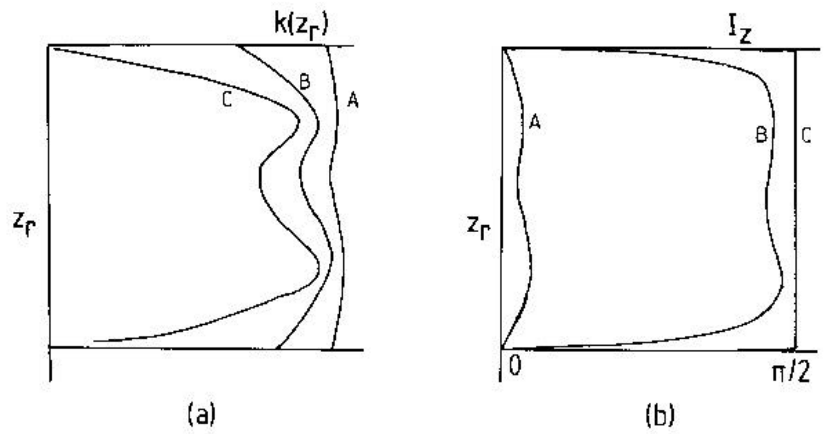

Fig. 1(a) Wavenumber depth profile for increasing velocity contrasts (A, B, C), and (b) the effect on the source-depth-averaged intensity I_z.

If trapping is perfect in the duct (for all effective ray angles) so that $k(z_0)=0$ then

$$I_2 = \pi/rH \tag{23}$$

the received intensity from a uniformly distributed set of sources in depth is independent of the depth and velocity profile.

The effects of varying velocity contrast can be seen from Eqn 21 or 22, and examples of low, intermediate and high contrast are shown diagrammatically in Fig 1. The intensity naturally falls to zero at the edge of the duct in all cases, and the shape of the intensity distribution looks more and more like the wavenumber distribution as the velocity contrast reduces.

Equation 22 can also be written in terms of the steepest permissible elevation angle at the receiver θ_r through

$$\cos\theta_r = k(z_0)/k(z_r) \tag{24}$$

and

$$I_2 = 2\phi_r/Hr \tag{25}$$

This shows that the identical formula quoted by Weston[10] for the special case of a uniform duct is, in fact, more general.

5. RANGE AVERAGE

Insertion of Eqn 17 into the incoherent sum formula, Eqn 3, and again converting to integral form yields

$$I_r = \frac{2}{\pi r}\int_{n=o}^{n=N} \frac{K\,(dK/dn)^2\,dn}{(k^2(z_r)-K^2)^{\frac{1}{2}}\,(k^2(z_s)-K^2)^{\frac{1}{2}}}$$

$$= \frac{2}{\pi r}\int_{k_{max}}^{k(z_o)} \frac{K\,(dK/dn)\,dK}{(k^2(z_r)-K^2)^{\frac{1}{2}}(k^2(z_s)-K^2)^{\frac{1}{2}}} \qquad (26)$$

The wavenumbers $k(z_s)$ *and* $k(z_r)$ *are the values at the source and* receiver depth and are just constants within the integral. The upper integral limit, k_{max}, is the minimum of these two, and it will be assumed below that this is $k(z_s)$; the opposite assumption would have been equally convenient, and the result can be seen by swapping s, r subscripts in the subsequent answers. The lower integral limit $k(z_0)$ is the wavenumber at the edge of the duct, as before.

To proceed any further with this formula dK/dn must be evaluated, and this depends on the velocity profile through the phase integral. Combining Eqns 15 and 11 gives a simple formula.

$$dK/dn = -\pi/(K \int (k^2(z) - K^2)^{-\frac{1}{2}}\,dz) \qquad (27)$$

At least three duct types give soluble solutions for this equation.

These are:

a) Uniform:

$$k(z)=k_1 \quad 0<z<H$$
$$k(z)=k(z_0) \quad z<0,\ z>H \qquad (28)$$

b) Linear:

$$k^2(z)=k_1{}^2(1-az) \qquad (29)$$

c) Parabolic:

$$k^2(z)=k_1{}^2(1-a^2z^2) \qquad (30)$$

where k_I and a are constants.

The solutions of Eqn 27 are easily shown to be for

a) Uniform:

$$dK/dn=-\pi(k_1{}^2-K^2)^{\frac{1}{2}}/KH \qquad (31)$$

where H is the water depth.

b) Linear:

$$dK/dn=-ak_1{}^2\pi/(2K(k_1{}^2-K^2)^{\frac{1}{2}}) \qquad (32)$$

c) Parabolic:

$$dK/dn=-ak_1/K \qquad (33)$$

These functions can be viewed as the wavenumber step between adjacent modes (ΔK=dK/dn), and their dependence on K is shown

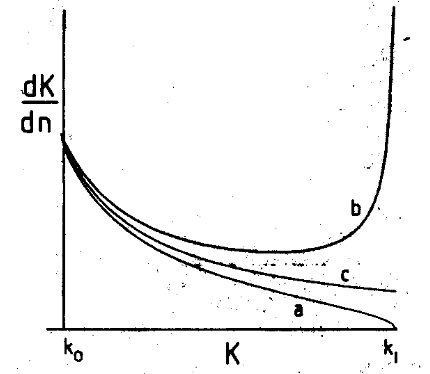

Fig. 2 Wavenumber (eigenvalue) spacing of modes for (a) uniform, (b) linear, and (c) parabolic ducts.

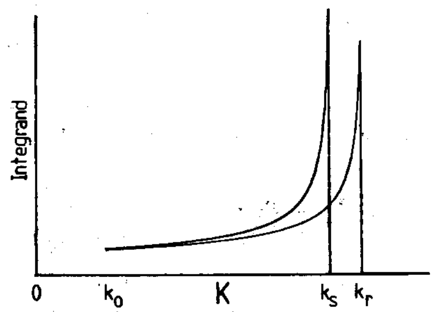

Fig. 3 Behaviour of the square root terms in the integrand for incoherent intensity (Eqn 26). Wavenumbers at source and receiver are k_s and k_r.

diagrammatically in Fig 2. For small K all behave as k_1/K, whereas for large K, dK/dn increases as k_1 is approached in a linear duct, but decrease in a uniform duct while staying nearly constant in a parabolic duct.

Remembering that K is proportional to the cosine of the grazing angle these observations are a restatement of the dependence of ray cycle distance r_c on ray angle [11] since

$$r_c = 2K \int (k^2(z) - K^2)^{-\frac{1}{2}} dz = -2\pi/(dK/dn) \quad (34)$$

Substituting Eqns 31, 32, or 33 into Eqn 26 yields soluble integrals for the three cases. The remaining two square root terms in Eqn 26 are shown diagrammatically in Fig 3.

5.1 Uniform Duct

Since $k(z_s)=k(z_r)=k_1$ Eqn 26 becomes

$$I_r = (2/Hr) \int_{k(z_0)}^{k_1} (k_1^2 - K^2)^{-\frac{1}{2}} dK$$

$$= (2/Hr) \cos^{-1} (k(z_0)/k_1) \quad (35)$$

In a uniform duct the maximum ray angle is related to wave number by $k(z_0)=k_1 \cos\theta_o$ so

$$I_r = 2\theta_o/rH \quad (36)$$

which agrees with simple acoustic flux arguments [5,10] as one would hope.

5.2 Linear Duct

$$I_r = (aK_1{}^2/r) \int_{k(z_0)}^{k(z_s)} \frac{dK}{(k^2(z_r)-K^2)^{\frac{1}{2}}(k^2(z_s)-K^2)^{\frac{1}{2}}(k_1{}^2-K^2)^{\frac{1}{2}}} \tag{37}$$

In the small angle approximation K is nearly constant $K \sim k_1$ and the integral can be written in the form

$$I_r = (\tfrac{1}{2}\, ak_1/r) \int_{X=k^2(z_0)}^{k^2(z_s)} \frac{dX}{(k^2(z)-X)^{\frac{1}{2}}(k^2(z_s)-X)^{\frac{1}{2}}(k_1{}^2-X)^{\frac{1}{2}}}$$

This is exactly the form obtained by Weston[5] using a more complex flux approach, and the solution is an elliptic integral [12] $F(p,q)$

$$I_r = (ak_1/r)\ (k_1{}^2 - k^2(z_s))^{-\frac{1}{2}}\ F(p,q) \tag{39}$$

where

$$p = \arcsin\left(\left[\frac{k^2(z_s)-k^2(z_0)}{k^2(z_r)-k^2(z_0)}\right]^{\frac{1}{2}}\right)$$

$$q = \left[\frac{k_1{}^2-k(z_r)^2}{k_1{}^2-k(z_s)^2}\right]^{\frac{1}{2}}$$

The multiplier in Eqn 39 is easily shown to be equivalent to Weston's $r^{-1}(2/Rz_s)^{\frac{1}{2}}$ where R is the ray's radius of curvature (a constant for low grazing angles). The integral has also been obtained by Brekhovskikh [13] using an alternative ray approach.

5.3 Parabolic Duct

$$I_r = (2ak_1/\pi r) \int_{k(z_0)}^{k(z_s)} \frac{dK}{(k^2(z_r)-K^2)^{\frac{1}{2}}(k^2(z_s)-K^2)^{\frac{1}{2}}} \tag{40}$$

In the small angle approximation with $K \sim k_1$, the integral can be written as

$$I_r = (a/\pi r) \int_{X=k^2(z_0)}^{k^2(z_s)} \frac{dX}{(k^2(z_r)-X)^{\frac{1}{2}}(k^2(z_s)-X)^{\frac{1}{2}}} \tag{41}$$

which again agrees with Weston's integral. The solution is

$$I_r = (a/\pi r)\ \ln\left(\frac{[((k^2(z_s)-k^2(z_0)^2)^{\frac{1}{2}}+(k^2(z_r)-k^2(z_0))^{\frac{1}{2}}]^2}{|(k^2(z_r)-k^2(z_s))|}\right) \tag{42}$$

which on making the substitution

$$k^2(z_0)/k_1{}^2 = 1-a^2z_0{}^2 = \cos^2\theta_0 \cong 1-\theta_0{}^2 \tag{43}$$

is identical with Weston's[4] formula.

6. NUMERICAL APPROACH

Application of these three analytical solutions may be somewhat limited, but the approach used may easily be applied numerically for any profile including multiple ducts, sea surface and sea bed. First, Eqn 27 is used to calculate dK/dn as a function of K by simply choosing K arbitrarily and tabulating. The water column can be split into layers in which $k^2(z)$ is linear

$$k^2(z)=k_i^{\,2}(1+a_i(z-z_i)) \tag{44}$$

(for the ith layer) giving a simple algebraic contribution to the integral for each layer after analytic integration of the form

$$\frac{2}{A}\left[(Az_{iT}+B)^{\frac{1}{2}}-(Az_{iB}+B)^{\frac{1}{2}}\right] \tag{45}$$

where $A=k_i^{\,2}a_i$

$B=k_i^{\,2}(1-a_iz_i)-K_n^{\,2}$

z_{iT}= depth at the top of the ith layer

z_{iB}= depth at the bottom of the ith layer

In more complex profiles (e.g. ones containing velocity maxima) care will need to be taken over the choice of relevant duct boundaries since this may change from one K region to the next. Also if the source and receiver are separated by a velocity maximum the effective upper limit in K for the integral must be the value of K at this velocity maximum.

Having tabulated the values of dK/dn and defined the limits in K it is then straightforward to integrate the smooth function in Eqn 26 numerically except for the region near the maximum K (i.e. $k(z_r)$ or $k(z_s)$) where there is usually an infinity in the integral. Provided source and receiver are not at conjugate depths ($k(z_s)\neq k(z_r)$) this can be handled by treating all terms in the integral as constant in a finite small region δK near k_{max} except for the term $(k_{max}^{\,2}-K^2)^{-\frac{1}{2}}$. This can be integrated analytically (giving $\cos^{-1}(\delta K/k_{max})$) and added to the remaining numerical integral from $k_{max}-\delta K$ down to $k(z_0)$.

7. CONCLUSIONS

Despite the fact that the above analytical solutions have been derived before, the WKB approach demonstrates that Weston's acoustic flux result (derived from a ray treatment) is equivalent to an incoherent sum over the discrete modes in the high frequency approximation.

The WKB approach is more versatile since numerical calculations are straightforward, and it is applicable to unaveraged as well as averaged intensity. Another advantage of the technique is that it can handle frequency dependence, and in fact the accompanying paper shows that the WKB method is adequate even when the number of modes is small provided the source and receiver are not at complementary depths or near a caustic. Indeed there is very little frequency dependence in the range average intensity under these conditions. Near the conjugate depth or a caustic the WKB method still works if a correction is made for the duct edges.

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