

The power of the equivalent source set as a criterion for the solution accuracy of the scattering problem

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ABSTRACT

The paper investigates the power of sources obtained with two variants of the Equivalent Source Method applied to an acoustic scattering problem: one using the least square method (*LSM*), the other, the full field equations (*FFE*). It is found that *LSM* and *FFE* produce source sets with totally different characteristics, *LSM* ones showing much more energetic interaction between sources than *FFE* ones. Quantitative results about the low total power a *LSM* appropriate monopole set should get are also given. These features are used to explain why, although much more unstable than *FFE*, *LSM* has a better ability to reproduce the problem boundary condition. Moreover, results obtained with a genetic algorithm show that, with *LSM*, this required low emitted (or absorbed) power should be produced by sources whose amplitudes are nevertheless “quite high”, these two criteria constituting some guidelines that ensure *LSM* accurate solutions.

Keywords: Scattering, equivalent sources, acoustic power, genetic algorithm

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1. INTRODUCTION

Sound field modelling, whether radiated by a vibrating structure or resulting from the scattering of an incident wave impinging on a body, is conventionally performed using the boundary element method. More recently developed, the *Equivalent Source Method* (*ESM*), simpler and therefore with a lower computational cost, is an interesting alternative. The idea of the method lies in the substitution of the body for a set of point sources – monopoles or multipoles - placed “inside” the body and whose complex amplitudes are determined so that the source set best reproduces, at the boundary, the velocity field corresponding to the real situation. However, the method presents two main drawbacks: while it does not always converge as the number of monopoles used in the set increases, the accuracy of the solution also depends strongly on the positioning of the sources, two factors intrinsically linked, as shown in [1].

In addition, there are no general rules for choosing an appropriate source set, even though different approaches have already been proposed to resolve the source positioning

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problem in some special cases (see, for instance, studies by Kropp and Svensson [2] or Koopmann and Fahnline [3]). Gounot and Musafir have investigated another approach [4] that aims to solve the problem in a global way. They have developed a computational code that combines *ESM* with genetic algorithms to simultaneously search for the amplitudes and positions of the sources responsible for the smallest possible boundary error. In this paper, solutions of a 3D scattering problem obtained with two variants of the *ESM* – one using the least square method (*LSM*), the other one being the “*Full Field Equations*” (*FFE*) proposed by Ochmann [5] – are compared. New results regarding the acoustic power of the equivalent sources (solutions of the problem) obtained with these two *ESM* variants are used in order to explain why, despite a significantly higher numerical instability, *LSM* can provide, in some cases, solutions with better precision.

2. THEORITICAL BACKGROUND

2.1 The scattering problem

When the surface of a body immersed in a fluid is vibrating with angular frequency ω , a sound pressure field p is radiated (see Figure 1a). In the frequency domain, p is the solution of the Neumann boundary value problem [6], namely p satisfies the Helmholtz equation (Equation 1)

$$\{\nabla^2 + k^2\} p(\mathbf{x}, \omega) = 0 \quad (1)$$

where $k = \omega / c_0$, c_0 being the sound propagation speed, and also the boundary condition given by Equation 2

$$\frac{\partial p(\mathbf{x}_S, \omega)}{\partial n} = -ik \rho_0 c_0 \bar{u}_n(\mathbf{x}_S, \omega), \quad (2)$$

where ρ_0 is the fluid mean density, \bar{u}_n is the normal velocity prescribed on the surface S , and \mathbf{x} and \mathbf{x}_S denote points in the propagation domain Ω_E and on the boundary, respectively.

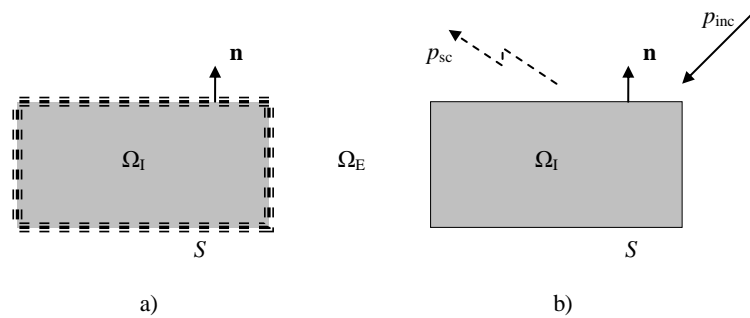


Figure 1: Representation of the radiation (a) and scattering (b) problems.

The scattering problem, generated by the impinging of an incident wave on a body can be seen and solved as a radiation problem (see Figure 1b), in which, now, the scattering pressure p_{sc} is solution of the Neumann boundary value problem. As for the boundary condition, if a rigid body is considered, the zero total normal velocity ($v_{sc} + v_n^{inc}$) on S , implies that the normal velocity prescribed on S must be thus $-v_n^{inc}$, i.e., the opposite of the velocity which would be generated “on S ” by the incident wave alone, in the absence of the body [1, 5].

2.2 The Equivalent Source Method

The *Equivalent Source Method* substitutes the real body for a set of M sources at points \mathbf{y}_m , placed strictly inside the body. The sound field due to these sources is expressed in terms of their unknown complex amplitudes, A_m , and of a function $g(\mathbf{x}, \mathbf{y})$ describing their radiation. Any function g that satisfies the Helmholtz equation and the radiation condition can be used. The simplest one, which corresponds to monopoles (used in this study), is the free-space Green function [6]. Expansions in spherical wave functions at one or various points are also frequently used. By virtue of the principle of superposition, the scattered pressure and velocity at \mathbf{x} can be written as

$$p_{sc}(\mathbf{x}) = \sum_{m=1}^M A_m g(\mathbf{x}, \mathbf{y}_m); \quad v_n^{sc}(\mathbf{x}) = \frac{-1}{i\omega\rho_0} \sum_{m=1}^M A_m \partial g(\mathbf{x}, \mathbf{y}_m) / \partial n \quad (3, 4)$$

Since a finite number of sources M is considered, the boundary condition cannot be exactly satisfied, and a local velocity error ε_v (given by the difference between the velocity due to the sources and the theoretical value $-v_n^{inc}$) is generated on S . For a given source set (i.e., a specified number and position), the “optimal” source strength $\{A_m^*\}$ are obtained by minimizing the velocity error on S and, in turn, Equation 3 provides an approximation for the scattered pressure field. The two *ESM* variants compared in this paper differ in the minimizing technique used: while *LSM* consists in minimizing the sum of the squared ε_v on the boundary, *FFE* uses the weighted residual technique in which the ponderation functions are the complex conjugate of the spherical wave functions (for more details, see [5]). With this latter variant, diagonally dominant matrices are generated, which are responsible for the *FFE* higher numerical stability.

3. NUMERICAL SIMULATION AND RESULTS

3.1 The Case under Study

The results presented in this work concern the 3D scattering problem given by the impinging of a plane wave on a rigid cube with dimension L equal to the wavelength of the incident wave (i.e., $kL = 2\pi$). The sources are monopoles which are uniformly distributed on “supports” with easy-to-implement geometries (one linear, parallel to the wave vector \mathbf{k} , and one circular, both centred at the centre of the cube). In order to investigate the influence of the source position and compactness on the solution accuracy, the size of the source supports is given by multiplying the cube dimension by a reduction factor a ($0 < a < 1$). The lower a is, smaller the source support, and consequently – for a given number of sources M – more compact and concentrated close to the cube centre will be the set. In order to evaluate the accuracy of the solutions obtained both with *LSM* and *FFE*, the normalised velocity error on the boundary, given by Equation 5, is used.

$$e_{BC} = \int_S |\varepsilon_v|^2 dS / \int_S |\bar{u}_n|^2 dS. \quad (5)$$

3.2 On the Emitted Acoustic Power

The acoustic power engendered by a sound source inside a closed surface can be expressed through the complex acoustic intensity, i.e. through the pressure and velocity fields radiated at the surface, given by

$$W_S = \frac{1}{2} \operatorname{Re} \left\{ \oint_S p \mathbf{v}^* \cdot \mathbf{n} dS \right\} = \frac{1}{2} \operatorname{Re} \left\{ \oint_S p v_n^* dS \right\}. \quad (6)$$

where $*$ denotes the complex conjugate, p and \mathbf{v} are the total pressure and velocity, v_n being the normal component of \mathbf{v} . In the case of scattering by a rigid body, the boundary condition being theoretically $v_n = 0$, the acoustic power is consequently zero (see Equation 6), since the scatterer obviously does not emit neither absorb any power, but merely ‘redistributes’, diffuses it in the exterior region.

Nevertheless, when the scatterer is modeled by a finite number of point sources – which is the case with *ESM* – the failure to reproduce exactly the boundary condition implies in a local velocity error field on S , ε_v , responsible for a net power emission (if positive value) or absorption (if negative value) given by

$$W_{ESM} = \frac{1}{2} \operatorname{Re} \left\{ \oint_S p \varepsilon_v^* dS \right\}. \quad (7)$$

This residual power produced by the equivalent sources can be directly computed from their source strengths [7], since the power emitted by a monopole of strength q_m located at \mathbf{y}_m is given by

$$W_m = (1/2\rho_0) \operatorname{Re} \left\{ p(\mathbf{y}_m) q_m^* \right\} \quad (8)$$

where $p(\mathbf{y}_m)$ is the total acoustic pressure at \mathbf{y}_m and $q_m = (i\omega)^{-1} A_m$. This procedure to obtain the source power is exact and will be used in the computation of W_{ESM} .

3.3 Results and Discussion

Figure 2 shows how characteristics of the *LSM* (in black) and *FFE* (in red) solutions vary as the number of monopoles used in the set increases from 2 to 50.

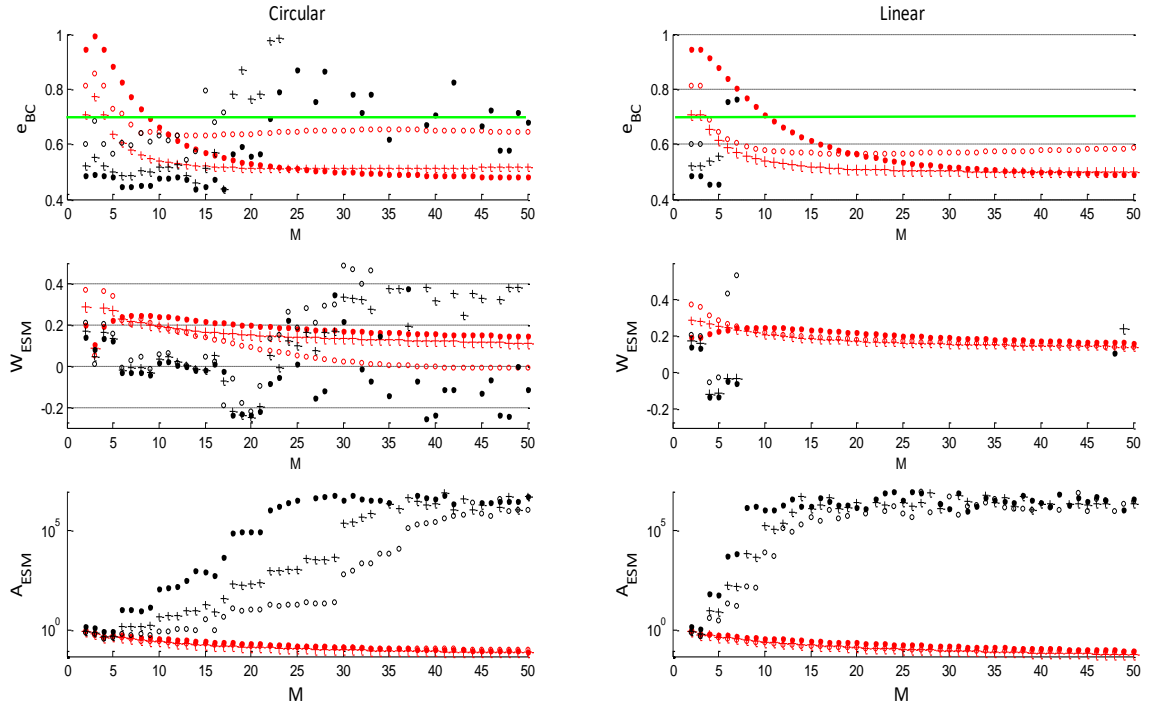


Figure 2: Boundary error, source set power and source strength mean amplitude for
a) circular and b) linear configurations versus the number of sources
with LSM (black) and with FFE (red) for $a = 0,25$ (\bullet), $a = 0,5$ ($+$) and $a = 0,75$ (\circ).

These characteristics are: the solution accuracy (e_{BC}), the source set normalized power (W_{ESM}) and its average source strength value (A_{ESM}) The results are shown for both circular and linear supports and, in each case, for three support sizes corresponding to $a = 0,25, 0,5$ and $0,75$.

First of all, one can observe the *FFE* stability and robustness, since the solution accuracy increases – namely, e_{BC} decreases – as the number of sources increases. This solution convergence (even though, slow) is obtained with source sets which show a slightly decrease in power and in average source amplitude as M increases. As for the influence of the source support (position of the sources), the best solutions are generally obtained with $a = 0.5$, i.e., when monopoles are neither too close to the border nor too concentrated gathered near the structure center.

As for the *LSM*, the e_{BC} curves show a somewhat chaotic aspect reflecting an important unstability, especially when M exceeds a certain threshold value (which intrinsically depends on the position of the sources), and beyond which the source amplitudes and powers may reach extremely high values. These rather high values are clearly propitious to engender numerical unstability and thus to cause a degradation of the solution accuracy, as shown in [1]. Nonetheless, maybe the most relevant result is that, below this M threshold value, *LSM* solutions are significantly more accurate than *FFE* ones. And, this is precisely because *LSM* can provide (more) precise solutions with very few sources – which represents a benefit not only numerical but also experimental if field reconstruction is envisaged – that we focus on what makes these *LSM* sources superior.

If considering only the *LSM* solutions with satisfactory accuracy (i.e., the ones with e_{BC} less than 0.7, a limit materialized by a green line in Fig. 2), it appears that they are all obtained by source sets with $|W_{ESM}| \leq 0.25$, namely, which total emitted (or absorbed) power is lower than 25% of the power received by the scatterer. Note that, although this condition seems necessary, it is obviously not sufficient, as evidenced by the following counterexample : as a matter of fact, even though a set made of exclusively zero amplitude sources would generate no power, it neither will produce any pressure and velocity field, and therefore cannot reconstruct any scattering problem boundary condition. Another main characteristic is that these low W_{ESM} values are observed despite a pretty high average source strength magnitude, up to 10^3 times the amplitudes of the sources provided by *FFE*. By the way, note that the discrepancy in the behavior observed between the circular and the linear supports (the solution degradation occurring for a much lower M value with the linear support) can be explained by the fact that, for a given M , the source set is much more dense and concentrated in a linear support than in a circular one, which allows a better spatial source distribution.

Figure 3 illustrates, for the 16 monopole circular set, the features of the individual sources (amplitude, phase and power) representative of the solutions that *FFE* and *LSM* commonly provide.

Regarding the source amplitudes, the results illustrate the recurrent fact that, in a given set, while *FFE* sources are very uniform (low and almost constant A_m values), *LSM* ones show a strong heterogeneity together with a wide range of values. As for the source phases, *FFE* and *LSM* sources show also features totally distinct: while *LSM* monopoles “organize themselves” into a sequence of local dipoles (2 juxtaposed monopoles being roughly in phase opposition), in the case of *FFE*, the ‘dipolar behaviour’ only appears globally, in the sense that the source set is divided into two source packs basically in phase opposition: in this particular case, one pack is made up of the monopoles #5-13

which are located in the “front” -relatively to the incident wave- of the circular support, while the other one is constituted by the other monopoles in the “rear”.

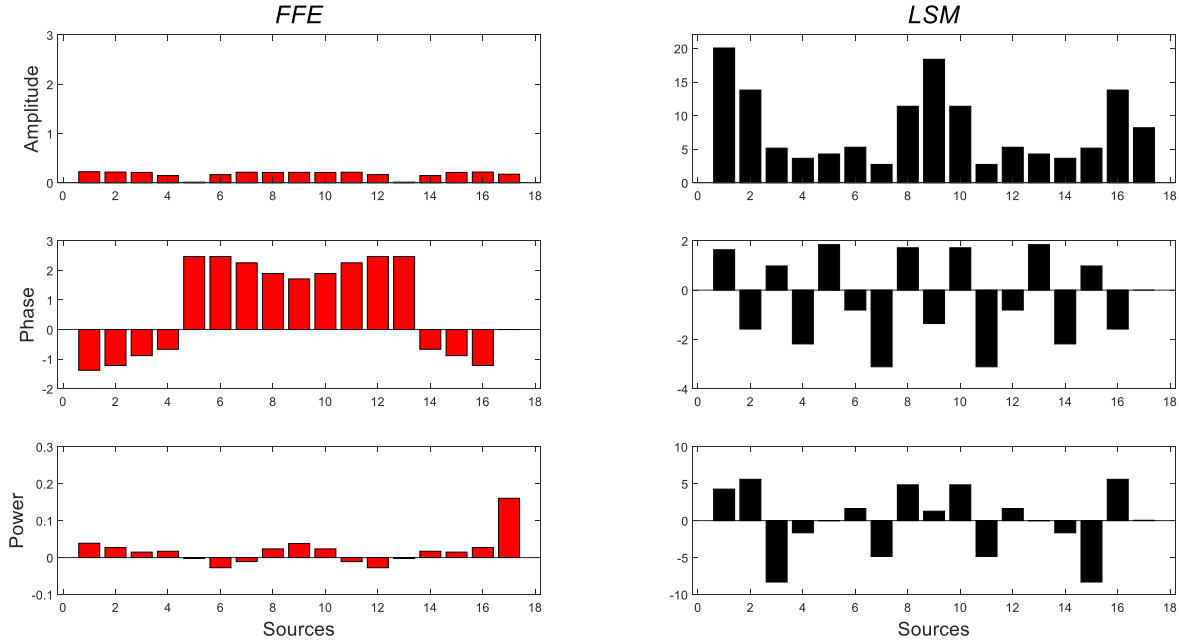


Figure 3: Amplitude, phase and power of the individual sources for the 16 monopole circular set obtained with a) FFE and b) LSM.

These amplitude and phase features mirror on the power of the individual sources. In the *LSM* case, there is a strong energetic interaction, some sources absorbing what others emit, which leads to a total power for the set (shown in the 17th column of the W_m graphs) always lower than the average of the individual powers. As for the *FFE* case, the interaction between sources is much smaller (most of the sources emit sound power) and the power of the source set is basically the sum of the individual powers. This larger energetic interaction that *LMS* induces is, in principle, a required characteristic for a better, more detailed reconstruction of the velocity field along the scatterer boundary while maintaining the lowest possible global power emission.

3.4 Further results using Genetic Algorithms

One way to get rid of the problematic positioning of the sources – mentioned in the anterior session – is to let the sources free to “move”. With this aim, a computational code called *ESGA* – which combines *ESM* with *Genetic Algorithms* –, has been developed and first presented in [4]. The code allows to find, for a given number of sources M , the “optimal” set, i.e. to determine simultaneously the best complex source strengths and position for the M monopoles responsible for the lower boundary error (e_{BC}) possible. Like all genetic algorithms, *ESGA* starts with the formation of a first population made of some possible solutions (called *chromosomes*) randomly generated. The fitness of each solution is evaluated, given by its corresponding value for the “cost function”, the function (e_{BC}) to be minimized. Three basic genetic operators - *selection*, *crossover* and *mutation* -, act on the first-generation chromosomes, preserving some of them and transforming others, and engendering a second generation, and so on. The *selection* operator makes that the best *chromosomes* in a population tend to survive to the next population, improving the quality of the successive generations. The *crossover* and *mutation* operators are responsible for the formation of new chromosomes, which

‘guarantees’ that the entire search space is scanned - actually it limits the risk of falling in a local minimum - and thus that the global optimal is found (for more details, see [8]).

The first line in Figure 4 shows four typical curves (obtained respectively with source sets made of $M = 2, 3, 4$ and 5 monopoles) of the cost function (e_{BC}) minimization, over 1000 *ESGA* generations/iterations. At each generation, the current best solution (i.e., the current best M source amplitudes and position) was stocked to allow, later, the computation of its average amplitude A_{ESM} and acoustic power W_{ESM} (2nd and 3rd lines). It is worth emphasizing that, in this process, the only function under minimization is the boundary error e_{BC} , and not the two other quantities A_{ESM} e W_{ESM} , afterward computed.

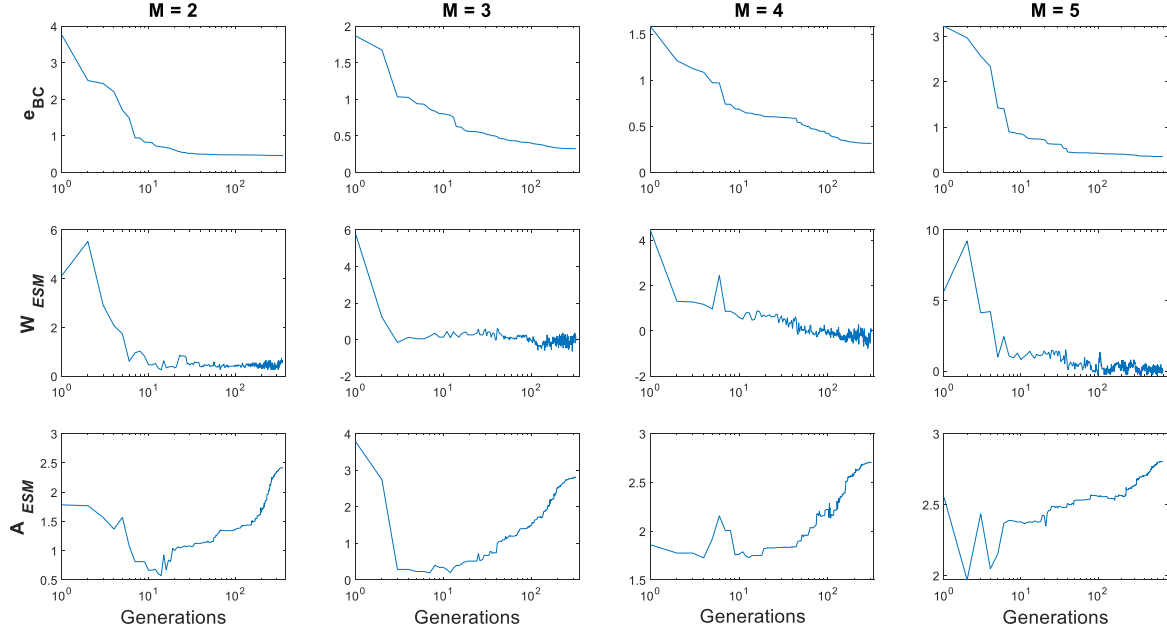


Figure 4: Boundary error (e_{BC}) minimized by *ESGA* (1st line) and set power (2nd line) and resulting source amplitude (3rd line) obtained for 2, 3 4 and 5 monopole sets.

All the numerical trials performed show a similar e_{BC} curve behavior, typical of the genetic algorithm minimization, in which an initial phase with a rapid decrease during the first generations is followed by a phase with a much slighter and slower one. Note that the e_{BC} values (under 0.40) obtained at the last generations - whether with 2, 3, 4 or 5 monopoles - are lower than the lowest ones obtained with up to 25 fix sources in a linear or a circular support (see Fig. 2), which confirms the potential of the *ESGA* technique. If we look at the power emitted by the source set found by the *ESGA*, one can notice that W_{ESM} curves mirror roughly these two e_{BC} curve phases: In the first minimization phase, W_{ESM} decreases quickly along with e_{BC} , the initial sets with very high power values (for the trials shown, the initial sources sets emit about 4 or 5 times the power received by the scatterer!) giving rapidly way to source sets with comparatively “negligible” power. In the second phase, as e_{BC} keeps on decreasing - even though very slowly, the source sets show power kept very low (a fraction of unity) and which can even oscillate between positive and negative values. Moreover, the results show that the values towards which the power tends (positive or negative) are compatible with the ones previously found (see Fig.2), with maximal absolute value about 25-30% of the power of the inciding wave.

Nevertheless, the most surprising result concerns the amplitude theses *ESGA* optimal source sets show. As in the case of e_{BC} (and W_{ESM}) curves, the average amplitude

curves always show two distinct phases. However, while in the first phase A_{ESM} curves generally decrease along with e_{BC} , during the second phase – in which the e_{BC} minimization is much slower, occurs an inversion in the shape of the A_{ESM} curves, their values starting to continuously increase. It is worth emphasizing that, independently of the number of monopoles used in the set, the A_{ESM} curves always converge to a similar specific value, between 2.5 and 3 in this case. Anyway, these W_{ESM} and A_{ESM} features tend to show that an appropriate source set should produce a low sound power (positive or negative), though, which should be engendered by sources with average amplitude relatively “high”, i.e., which are able to cause important energetic interaction between them, a result that confirms the one evidenced with “fixed” sources in Figure 3.

4. CONCLUSIONS

Focused on the features of the equivalent source strengths, this investigation contributes to a better understanding of the reasons why, beside a strong instability, LSM can lead - especially when the number of sources used is small - to solutions with better accuracy than FFE . Results concerning individual sources have shown that this benefit is due to the fact that, unlike FFE sources, LSM ones show a significative energetic interaction – which allows a finer and more detailed reconstruction of the boundary condition. Quantitatively, it was verified that all the source sets that provide accurate solutions present a total power (emitted or absorbed) less than 25% of the power received by the scatterer, which is obviously a necessary but not sufficient condition. Furthermore, results obtained by using the genetic algorithm $ESGA$ have highlighted the fact that the solution accuracy is conditioned not only to this low total power, but also to the fact that this low power must be generated by sources with relatively “high” amplitudes. Besides as guidelines, these source set criteria could be benefitted in a future work, – by including them as penalties in the aptitude function to be minimized – in order to accelerate the convergence of the $ESGA$ algorithm.

5. ACKNOWLEDGEMENTS

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6. REFERENCES

1. Y. J. R. Gounot, R. E., Musafir and J. G. Slama. *A comparative study of two variants of the equivalent sources method in scattering problems*, *Acustica United with Acta Acustica*, 91, 860-872. (2005).
2. W. Kropp and P.U. Svensson. *Application of the time domain formulation of the method of equivalent sources to radiation and scattering problems*, *Acustica*, 81, 528-543 (1995).
3. L. Song, G.H. Koopmann, J.B. Fahline, *Numerical errors associated with the method of superposition for computing acoustic fields*, *J. Acoust. Soc. Am.*, 89, 2625-2633. (1991).
4. Y. J. R. Gounot and R. E. Musafir. *Genetic Algorithms: a global search tool to find optimal equivalent source sets*, *Journal of Sound and Vibration*, 322, 282-298. (2009).
5. M. Ochmann, “*The Full-Field Equations for Acoustic Radiation and Scattering*”. *J. Acoust. Soc. Am.* 105(5), 2574-2584 (1999)
6. A.D. Pierce, *Acoustics. An introduction to its physical principles and applications*. New York, McGraw-Hill Book Company (1991)
7. P.A. Nelson, S.J. Elliott, *Active control of sound*, Academic Press (1992)
8. D.E. Goldberg: *Genetic Algorithms in search, optimization, and machine learning*. Massachusetts: Addison-Wesley (1989).