MADRID
inter.noise 2019
June 16 - 19
NOISE CONTROL FOR A BETTER ENVIRONMENT

Vibration Transmissibility of Two-DOF Vibrating Systems

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ABSTRACT

In most engineering applications, the vibration isolation problem, or vibration transmissibility, is studied using the one-DOF model due to its simplicity. In reality, however, if the vibration isolation system is mounted on a non-rigid structure or foundation, then it leads to a two-DOF vibrating system. A quite common case is a piece of rooftop equipment, such as fans and cooling towers, mounted on either springs or neoprene pads, because the roof structure itself is flexible and introduces the second degree of freedom. Occasionally, a piece of mechanical equipment is both internally and externally isolated which is also a two-DOF vibrating system. A floating floor and the structural floor also make a two-DOF system. This paper discusses the theoretical models of vibration transmissibility of two-DOF vibrating systems. A literature review is provided and errors in some references are corrected. Finite-element-models are developed to verify the theoretical analysis.

Keywords: Two-DOF, Vibration Transmissibility, Vibration Isolation
I-INCE Classification of Subject Number: 46

1. INTRODUCTION

Vibration transmissibility (VT) includes two types of problems: (1) response of a vibrating system due to the base excitation, and (2) vibration isolation. The purpose of vibration isolation is to reduce the vibration energy transmitted from a vibrating system to the structure or foundation, which further reduces possible damage to the structure as well as the acoustical radiation of the structure.

Various vibration isolation systems, such as neoprene pads, springs, dashpot dampers, composite high damping materials, etc. have been successfully used in different applications. In most engineering applications, the vibration transmissibility is modelled using the one-DOF (degree-of-freedom) model due to its simplicity. By assuming viscous damping, the vibration transmissibility is given by

$$\mathrm{VT}_{\text{1-DOF}} = \frac{\sqrt{1+(2\zeta r)^2}}{\sqrt{\left(1-r^2\right)^2+(2\zeta r)^2}}, \tag{1}$$

where r is the excitation frequency to natural frequency ratio ω/ω_n, and ζ is the damping ratio. The goal of vibration isolation is to keep VT < 1.

In reality, however, many vibration transmissibility problems should be modelled as two-DOF, or even higher DOF, systems. For example, a vibration isolation

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system is mounted on a non-rigid structure or foundation. A quite common case is a piece of rooftop equipment, such as fans and cooling towers, mounted on either springs or neoprene pads, because the roof structure itself is flexible and introduces the second degree of freedom. Occasionally, a piece of mechanical equipment is both internally and externally isolated which is a two-DOF vibrating system. For example, fans and motors in an air-handling-unit are internally isolated and the whole unit is externally isolated as well. A floating floor along with the structural floor also makes a two-DOF system.

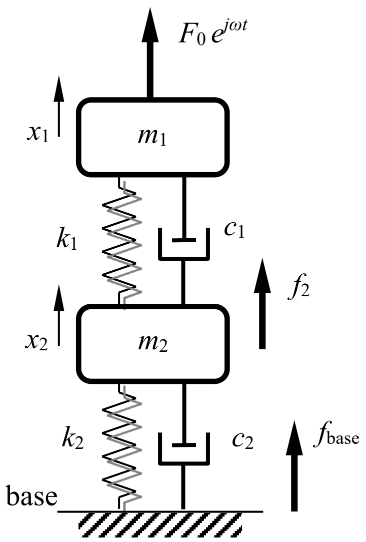

Figure 1. Model of two-DOF vibrating system.

A two-DOF vibrating system is illustrated in Figure 1. Two masses are m_1 and m_2, respectively. k_1 and k_2 are stiffnesses. In this paper viscous damping is assumed. An external harmonic force $f_1 = F_0 e^{j\omega t}$ is applied to m_1. The two displacement responses are x_1 and x_2 in the time domain. The external force f_1 is transmitted to m_2 and base, such that $f_2 = k_1 \left(x_1 - x_2\right) + c_1 \left(\dot{x}_1 - \dot{x}_2\right)$ and $f_{\text{base}} = k_2 x_2 + c_2 \dot{x}_2$. Therefore, there are two vibration transmissibilities: $F_2 \left(\omega\right) / F_1 \left(\omega\right)$ and $F_{\text{base}} \left(\omega\right) / F_1 \left(\omega\right)$ in the frequency domain. If m_1 is the vibrating machine to be isolated and m_2 is the non-rigid structure, then $F_2 \left(\omega\right) / F_1 \left(\omega\right)$ is the main concern. In this paper the two-DOF vibration transmissibility is defined as

$$\mathrm{VT}_{\text{2-DOF}} = \frac{F_2 \left(\omega\right)}{F_1 \left(\omega\right)}. \tag{2}$$

The theoretical models of vibration transmissibility of two-DOF systems have been studied by many researchers. Soliman and Hallam studied the vibration isolation on non-rigid foundations [1]. They derived the formula of vibration transmissibility of undamped two-DOF systems. Although they presented plots of vibration transmissibility with three different damping ratios, they did not derive the theoretical formula and they assumed a very special case $k_1 = k_2$ and $c_1 = c_2$ in those plots. It is also worth mentioning that in this 1968's paper, the critical damping was defined differently $c_c = 2\sqrt{2mk}$, while nowadays the critical damping is defined as $c_c = 2\sqrt{mk} = 2m\omega_n$. The ASHRAE Handbook includes the formula of two-DOF vibration transmissibility [2]. However, the formula does not include damping, either. Rivin derived the second vibration transmissibility $F_{\text{base}}(\omega)/F_1\ (\omega)$, again, for undamped two-DOF systems [3] (pp.48-61).

Many researchers studied tuned-mass damper and the effects on vibration transmissibility [4] - [6]. A main vibrating system with a turned-mass damper is a two-DOF system. However, the configuration is different than Figure 1. In Rusicka's model, m_1 is the added turned-mass damper while m_2 is the vibrating machine [5]. The driving force is f_2 and the foundation was assumed rigid, so the vibration transmissibility is actually $F_{\text{base}}(\omega)/F_2\ (\omega)$.

Unfortunately, a literature review did not find the theoretical formula of damped two-DOF vibration transmissility. In addition, due to the complexity of such a two-DOF problem some researchers made mistakes or misused some of the formulas.

For example, in the first edition of the well-known *Noise and Vibration Control* edited by Beranek and Vér, Dr. Eric Ungar presented the formula of undamped two-DOF vibration transmissibility (Eq. 11.10 on page 438) [7]. There is a misprint regrettably, and the same misprint remains in the second edition of 2006 (Eq. 13-10 on page 566) [8]. Fortunately, the formula to calculate two resonance frequencies and the associated figure are correct. The details are explained in Section 2.2 of this paper.

Waters and Sherren discussed the design of floating floor [9]. They cited a formula from Reference [6] and treated it as "vibration transmissibility". However, the formula is actually the magnification factor of a vibrating system with a tuned-mass damper $\left|\dfrac{X}{F/k}\right|$, where F is the force applied to the main mass (m_2 in Figure 1) and X is the displacement response of the same mass. In addition, they sadly copied the formula wrong where the right bracket in the denominator's first term was placed at wrong position.

In this paper the theoretical formula of damped two-DOF vibration transmissibility is derived for the first time. Different cases are presented with detailed discussions. A finite element model is developed in ANSYS to verify the theoretical analyses. A case study of floating floor design is also presented.

2. THEORETICAL MODELS

2.1. Derivation of the theoretical formula with damping

Equations of motion for a damped two-DOF vibrating system shown in Figure 1 can be easily obatined:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix}+\begin{bmatrix} c_1 & -c_1 \\ -c_1 & c_1+c_2 \end{bmatrix}\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix}+\begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1+k_2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}=\begin{Bmatrix} F_0\, e^{j\omega t} \\ 0 \end{Bmatrix}. \tag{3}$$

If we assume harmonic solutions $\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}=\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}e^{j\omega t}$ and plug back in Eq. (3), we have

$$\underbrace{\begin{bmatrix} k_1-m_1\omega^2+j\omega c_1 & -k_1-j\omega c_1 \\ -k_1-j\omega c_1 & (k_1+k_2)-m_2\omega^2+j\omega(c_1+c_2) \end{bmatrix}}_{\mathbf{Z}}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}=\begin{Bmatrix} F_0 \\ 0 \end{Bmatrix}, \tag{4}$$

where $\mathbf{Z}$ is the impedance matrix. Its determinant can be calculated as below.

$$\begin{aligned}\det(\mathbf{Z}) &= \left(k_1 - m_1\omega^2 + j\omega c_1\right)\left[\left(k_1 + k_2\right) - m_2\omega^2 + j\omega\left(c_1 + c_2\right)\right] - \left(k_1 + j\omega c_1\right)^2 \\ &= \left\{k_1 k_2 - \left[k_1 m_2 + \left(k_1 + k_2\right)m_1 + c_1 c_2\right]\omega^2 + m_1 m_2 \omega^4\right\} \\ &\quad + j\omega\left[k_2 c_1 + k_1 c_2 - \omega^2\left(m_2 c_1 + m_1 c_1 + m_1 c_2\right)\right] .\end{aligned} \tag{5}$$

In the frequecny domain, the force transmitted to m_2 is

$$F_2(\omega) = \left(X_1 - X_2\right)\left(k_1 + j\omega c_1\right), \tag{6}$$

where the two displacements can be calculated by using the Cramer's rule:

$$\begin{cases} X_1 = \dfrac{F_0\left[\left(k_1 + k_2\right) - m_2\omega^2 + j\omega\left(c_1 + c_2\right)\right]}{\det(\mathbf{Z})} \\ X_2 = \dfrac{F_0\left(k_1 + j\omega c_1\right)}{\det(\mathbf{Z})} \end{cases} . \tag{7}$$

Plug Eqs. (5) and (7) in Eq. (6), then the force transmitted to m_2 is

$$\begin{aligned}F_2(\omega) &= \frac{F_0}{\det(\mathbf{Z})}\left[\left(k_1 + k_2\right) - m_2\omega^2 + j\omega\left(c_1 + c_2\right) - \left(k_1 + j\omega c_1\right)\right]\left(k_1 + j\omega c_1\right) \\ &= \frac{F_0}{\det(\mathbf{Z})}\left[k_1 k_2 - \left(k_1 m_2 + c_1 c_2\right)\omega^2 + j\omega\left(k_2 c_1 + k_1 c_2 - m_2 c_1 \omega^2\right)\right]\end{aligned} . \tag{8}$$

The two-DOF vibration transmissibility is then

$$\begin{aligned}\mathrm{VT}_{\text{2-DOF}} &= \left|\frac{F_2}{F_0}\right| = \frac{\left|k_1 k_2 - \left(k_1 m_2 + c_1 c_2\right)\omega^2 + j\omega\left(k_2 c_1 + k_1 c_2 - m_2 c_1 \omega^2\right)\right|}{\left|\det(\mathbf{Z})\right|} \\ &= \frac{\sqrt{\left[k_1 k_2 - \left(k_1 m_2 + c_1 c_2\right)\omega^2\right]^2 + \omega^2\left(k_2 c_1 + k_1 c_2 - m_2 c_1\right)^2}}{\sqrt{\begin{aligned}&\left\{k_1 k_2 - \left[k_1 m_2 + \left(k_1 + k_2\right)m_1 + c_1 c_2\right]\omega^2 + m_1 m_2 \omega^4\right\}^2 \\ &+ \omega^2\left[k_2 c_1 + k_1 c_2 - \omega^2\left(m_2 c_1 + m_1 c_1 + m_1 c_2\right)\right]^2\end{aligned}}}\end{aligned} . \tag{9}$$

Let $\omega_1 = \sqrt{\dfrac{k_1}{m_1}}$, $\omega_2 = \sqrt{\dfrac{k_2}{m_2}}$, $\beta = \dfrac{\omega_1}{\omega_2}$, $\mu = \dfrac{m_1}{m_2}$, $\zeta_1 = \dfrac{c_1}{2\sqrt{m_1 k_1}}$, $\zeta_2 = \dfrac{c_2}{2\sqrt{m_2 k_2}}$, $r = \dfrac{\omega}{\omega_1}$.

Dividing both the numerator and denomintor by $k_1 k_2$ and after lengthy simplification, the theoretical formula of damped two-DOF vibration transmissibility is obtained:

$$\mathrm{VT}_{\text{2-DOF}} = \left|\frac{F_2}{F_0}\right| = \frac{\sqrt{\left[1 - \left(\beta^2 + 4\zeta_1\zeta_2\beta\right)r^2\right]^2 + 4r^2\left(\zeta_1 + \zeta_2\beta - \zeta_1\beta^2 r^2\right)^2}}{\sqrt{\begin{aligned}&\left\{1 - \left[(1+\mu)\beta^2 + 1 + 4\zeta_1\zeta_2\beta\right]r^2 + \beta^2 r^4\right\}^2 \\ &+ 4\left\{\left(\zeta_1 + \zeta_2\beta\right)r - \left[\zeta_1(1+\mu)\beta^2 + \zeta_2\beta\right]r^3\right\}^2\end{aligned}}} . \tag{10}$$

While Eq. (9) includes six parameters (m_1, m_2, k_1, k_2 c_1, and c_2), Eq. (10) only has four parameters (μ, β, ζ_1, and ζ_2).

Similarly, the force transmitted to base is

$$F_{\text{base}}(\omega) = X_2\left(k_2 + j\omega c_2\right) = \frac{F_0\left(k_1 + j\omega c_1\right)\left(k_2 + j\omega c_2\right)}{\det(\mathbf{Z})} \;. \tag{11}$$

Then the second vibration transmissibility is

$$\left|\frac{F_{\text{base}}}{F_0}\right| = \frac{\sqrt{\left[k_1 k_2 - c_1 c_2 \omega^2\right]^2 + \omega^2\left(k_2 c_1 + k_1 c_2\right)^2}}{\left|\det(\mathbf{Z})\right|} \;, \tag{12}$$

which can be further simplified as

$$\left|\frac{F_{\text{base}}}{F_0}\right| = \frac{\sqrt{\left[1-4\zeta_1\zeta_2\beta r^2\right]^2 + 4r^2\left(\zeta_1+\zeta_2\beta\right)^2}}{\sqrt{\begin{array}{l}\left\{1-\left[(1+\mu)\beta^2+1+4\zeta_1\zeta_2\beta\right]r^2+\beta^2 r^4\right\}^2\\ +4\left\{\left(\zeta_1+\zeta_2\beta\right)r-\left[\zeta_1(1+\mu)\beta^2+\zeta_2\beta\right]r^3\right\}^2\end{array}}} \;. \tag{13}$$

This paper focuses on the two-DOF vibration transmissibility given by Eq. (10). m_1 is the vibrating machine to be isolated and m_2 is the non-rigid structure. A few cases and details are discussed in the following sections. Eq. (13) can be used in other appropriate applications. For example, the vibration of a parking garage (base in Figure 1) is transmitted through columns (k_2 and c_2) to the structural floor above (m_2) and further to the system (m_1) that is to be isolated. Although the actual situation is more complicated since the bending vibration of columns should also be included in the parking garage vibration, such a simplified model can approximate the vibration isolation in the vertical direction to a certain extent.

2.2. Case A: $\zeta_1 = \zeta_2 = 0$

This is the simplest case where damping for both degrees of freedom is ignored. Vibration transmissibility Eq. (10) is simplified as

$$\left|\frac{F_2}{F_0}\right|_{\zeta_1=\zeta_2=0} = \frac{\left|1-\beta^2 r^2\right|}{\left|1-\left[(1+\mu)\beta^2+1\right]r^2+\beta^2 r^4\right|} \;. \tag{14}$$

As explained in Section 1. Introduction, a few references derived this formula.

Remark #1. Obviously, when the frequency ratio $r = 1/\beta = \omega_2/\omega_1$, in another word, when the excitation frequency is $\omega = \omega_2$, the numerator of Eq. (14) becomes zero. It means the vibration energy does not transmit to the non-rigid structure at this specific frequency. Therefore, the system can be tuned accordingly in order to achieve very low vibration transmissibility.

Remark #2. The two resonance frequencies are where the transmissibility approaches infinity. Unlike the damped systems, there are no local maxima in Eq. (14), so setting the derivative of Eq. (14) to zero will not lead to real roots. Instead, the two resonance frequencies are calculated by setting the denominator of Eq. (14) to zero:

$$\beta^2 r^4 - \left[(1+\mu)\beta^2+1\right]r^2+1=0 \Rightarrow r_{r_1,r_2}^2 = \frac{(1+\mu)\beta^2+1\pm\sqrt{(1+\mu)^2\beta^4+2(\mu-1)\beta^2+1}}{2\beta^2}.$$

Notice the quantity under the square root is a quadratic function of β^2. The coerfficient of β^4 is positive, and the discriminant is $\Delta = 2^2(\mu-1)^2-4(\mu+1)^2<0$. Therefore, the quantity of square root is always positive and less than $(1+\mu)\beta^2+1$. In another word, there are two distinct positive roots:

$$r_{r_1,r_2} = \sqrt{\frac{(1+\mu)\beta^2+1\pm\sqrt{(1+\mu)^2\beta^4+2(\mu-1)\beta^2+1}}{2\beta^2}}. \tag{15}$$

It is interesting to observe that the smaller root is less than unity and the larger root is greater than $1/\beta$. It implies that the resonance frequencies of a two-DOF vibrating system are lower than ω_1 and higher than ω_2, respectively. When μ is small, the two roots are very close to unity and $1/\beta$, which means the two resonance frequencies are very close to each individual DOF's natural frequencies ω_1 and ω_2.

In addition, in order to study the effect of mass ratio μ on the two resonance frequencies, take the derivative of Eq. (15) with respect to μ:

$$\frac{dr_{r_1,r_2}}{d\mu} = \frac{1\pm\dfrac{(1+\mu)\beta^2+1}{\sqrt{\left[(1+\mu)\beta^2+1\right]^2-4\beta^2}}}{4r_{r_1,r_2}}. \tag{16}$$

It is easy to show that for r_{r_1} (the one with negative sign), $dr_{r_1}/d\mu<0$. So the first resonance frequency decreases with an increase in μ. On the other hand, $dr_{r_2}/d\mu>0$ (the one with positive sign), so the second resonance frequency increases with an increase in μ. Therefore, the two resonance frequencies would further separate (the smaller one gets smaller and the larger one gets larger) when the mass ratio $\mu = m_1/m_2$ increases.

Remark #3. It was mentioned in Section 1. Introduction that there is a misprint in References [7] and [8]. The vibration transmissibility is printed as $T_F = \left|\dfrac{1-R^2}{(1-R^2)(1-R^2G^2)-R^2/M}\right|$, where $R=\dfrac{\omega}{\omega_2}$, $G=\dfrac{\omega_2}{\omega_1}$, and $M=\dfrac{m_1}{m_2}$. A careful comparison with Eq. (14) indicates that the second term of the demoninator should be R^2M. So the correct formula should be

$$T_F = \left| \frac{1-R^2}{\left(1-R^2\right)\left(1-R^2G^2\right)-MR^2} \right|.$$

2.3. Case B: $\zeta_2 = 0$

If the non-rigid structure's damping is negligible, $\zeta_2 = 0$, then the vibration transmissibility Eq. (10) can be simplified as

$$\left|\frac{F_2}{F_0}\right|_{\zeta_2=0} = \frac{\sqrt{\left[1-\beta^2r^2\right]^2+4\left(\zeta_1 r-\zeta_1 \beta^2 r^3\right)^2}}{\sqrt{\left\{1-\left[(1+\mu)\beta^2+1\right]r^2+\beta^2r^4\right\}^2+4\left[\zeta_1 r-\zeta_1 (1+\mu)\beta^2r^3\right]^2}} = \frac{\sqrt{\left(1-\beta^2r^2\right)^2\left(1+4\zeta_1^2r^2\right)}}{\sqrt{\left\{1-\left[(1+\mu)\beta^2+1\right]r^2+\beta^2r^4\right\}^2+4\zeta_1^2r^2\left[1-(1+\mu)\beta^2r^2\right]^2}}. \tag{17}$$

Figure 2 illustrates vibration transmissibility curves of a specific μ-β combination: $\mu = 2$, $\beta = 0.5$ with different damping ratios. Some interetsting observations and remarks are discussed as followes.

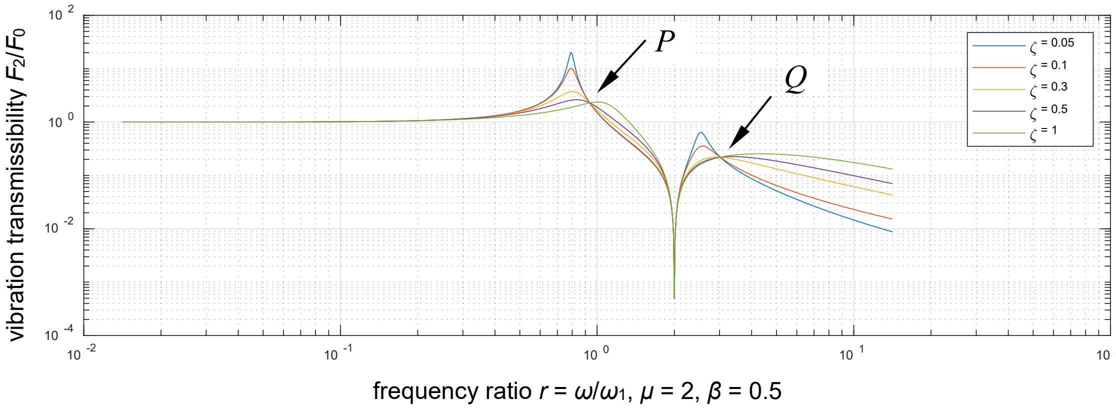

Figure 2. Vibration transmissibility curves of $\mu = 2$, $\beta = 0.5$.

Remark #1: Similar to Remark #1 of the last section, when $r = 1/\beta = \omega_2/\omega_1$, the numerator of Eq. (17) becomes zero, then the transmissibility is zero. This is independent of damping. This is shown in Figure 2 where all the curves approach zero at r = 2. Therefore, very effective vibration isolation can be obtained at frequencies close to ω_2 .

Remark #2: It is quite interesting to observe that all the curves cross two points P and Q, independent of the damping ratio. These points are similar to the point at $r = \sqrt{2}$ for the one-DOF vibration transmissibility. These two points can be solved as follows.

Inspired by Den Hartog's research on tuned mass damper [4] (pp. 93-106), Eq. (17) can be re-written as

$$\left|\frac{F_2}{F_0}\right|_{\zeta_2=0} = \frac{\sqrt{A^2+A^2\left(4\zeta_1^2r^2\right)}}{\sqrt{B^2+C^2\left(4\zeta_1^2r^2\right)}}, \tag{18}$$

where $A=1-\beta^2r^2$, $B=1-\left[(1+\mu)\beta^2+1\right]r^2+\beta^2r^4$, and $C=1-(1+\mu)\beta^2r^2$. The vibration transmissibility is independent of damping ratio ζ_1 when $\frac{A^2}{B^2}=\frac{A^2}{C^2}$. There are two possibilities.

Possibility #1: $B=C\Rightarrow\beta^2r^4-r^2=0\Rightarrow r=0$ or $r=1/\beta$. Obviously these are not points *P* and *Q*.

Possibility #2: $B=-C\Rightarrow\beta^2r^4-\left[2(1+\mu)\beta^2+1\right]r^2+2=0$

$$\Rightarrow r^2=\frac{2(1+\mu)\beta^2+1\pm\sqrt{4(1+\mu)^2\beta^4+4(\mu-1)\beta^2+1}}{2\beta^2}.$$

Notice the quantity under the square root is a quadratic function of β^2. The discriminant is $\Delta=16(\mu-1)^2-16(\mu+1)^2<0$. Therefore, the quantity of square root is always positive and less than $2(1+\mu)\beta^2+1$. In another word, there are two distinct positive solutions:

$$r_{P,Q}=\sqrt{\frac{2(1+\mu)\beta^2+1\pm\sqrt{4(1+\mu)^2\beta^4+4(\mu-1)\beta^2+1}}{2\beta^2}}. \tag{19}$$

For $\mu=2$ and $\beta=0.5$, the two points are at $r_P=0.9364$ and $r_Q=3.0204$, which agree with Figure 2.

Remark #3: To find the two resonance frequencies, differentiate Eq. (17) and set $\frac{d}{dr}\left|\frac{F_2}{F_0}\right|_{\zeta_2=0}=0$. After quite rigorous algebra, the two resonance frequencies can be obtained by solving a 10th order polynomial:

$$\begin{aligned}
&2\beta^6\zeta_1^2r^{10}+\left(\beta^6-6\beta^4\zeta_1^2\right)r^8\\
&-\left\{\beta^6(\mu+1)+\beta^4\left[16\mu\zeta_1^4(\mu+1)-4\mu\zeta_1^2+3\right]-6\beta^2\zeta_1^2\right\}r^6\\
&+\left\{\beta^4(\mu+1)\left(3-8\mu\zeta_1^2\right)+\beta^2\left[4\mu\zeta_1^2\left(4\zeta_1^2-1\right)+3\right]-2\zeta_1^2\right\}r^4\\
&-\left[\beta^4\mu(\mu+1)-\beta^2\left(8\mu\zeta_1^2-2\mu-3\right)+1\right]r^2+\mu\beta^2+1=0\ .
\end{aligned} \tag{20}$$

For $\mu=2$ and $\beta=0.5$, the resonance frequencies of different damping ratios are listed in Table 1. These values are the positive real roots of such a 10th order polynomial.

Table 1. Resonance frequencies of the system shown in Figure 2.

	r_{p1}	r_{p1}
$\zeta_1=0$	0.7923	2.5243
$\zeta_1=0.05$	0.7925	2.5370
$\zeta_1=0.1$	0.7930	2.5754
$\zeta_1=0.3$	0.8022	2.9263
$\zeta_1=0.5$	0.8336	3.3684
$\zeta_1=1$	1.0072	4.3099

Figure 3 shows a few cases of μ-β combinations. Each plot includes various damping ratios.

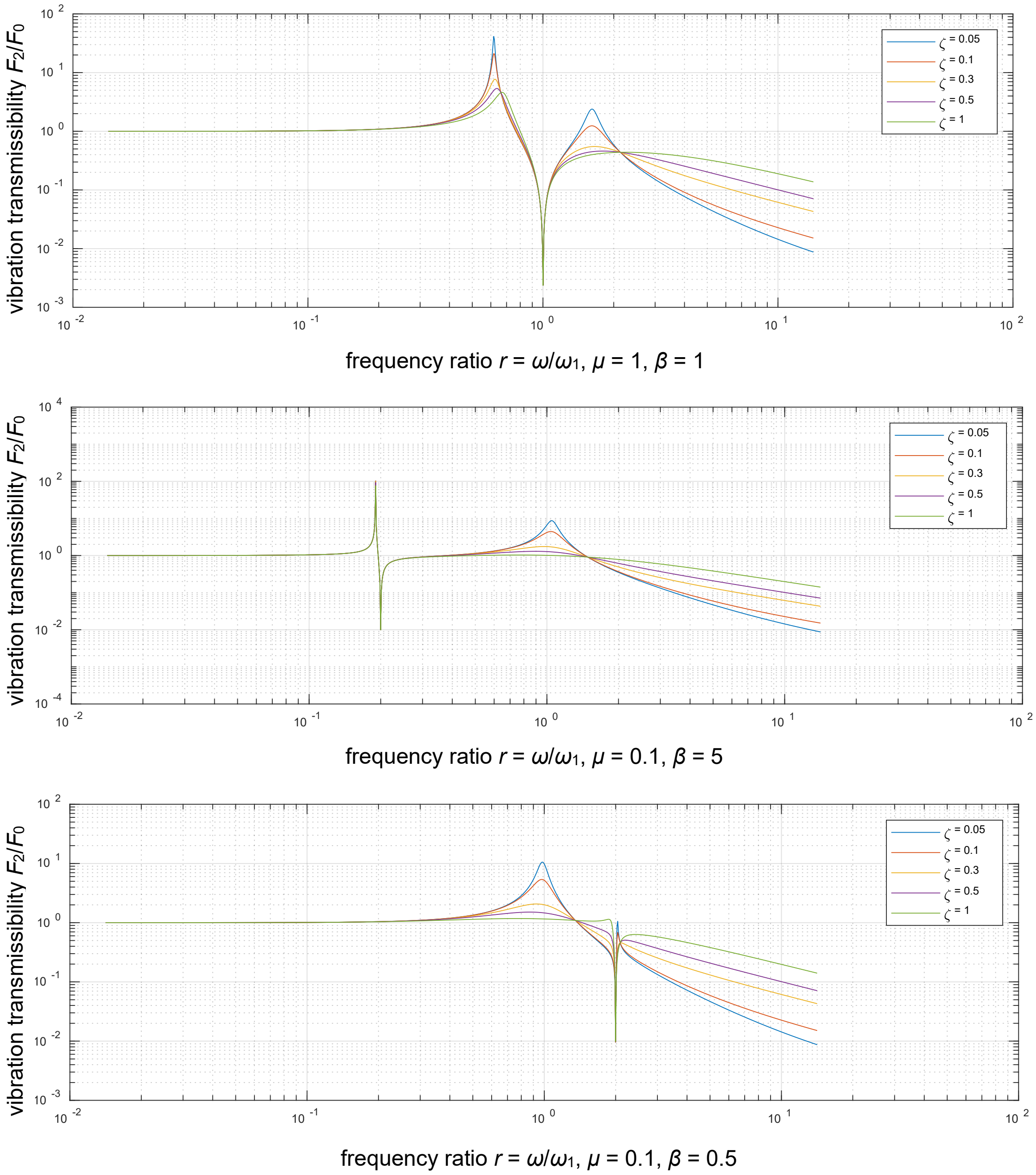

Figure 3. Vibration transmissibilities of different μ and β combinations.

Remark #4: Figure 4 illustrates the effect of the mass ratio μ. Similar to Remark #2 of Section 2.2 (see Eq. (16)), as μ increases, the two resonance frequencies are more separated in frequency, leaving a wide frequency range of low vibration transmissibility between the two peaks. On the other hand, if $\mu \to 0$, denominator can be simplified as

$$\begin{aligned}
&\sqrt{\left\{1-\left[\left(1+\mu\right)\beta^2+1\right]r^2+\beta^2 r^4\right\}^2+4\zeta_1^2 r^2\left[1-\left(1+\mu\right)\beta^2 r^2\right]^2} \\
&=\sqrt{\left(1-\beta^2 r^2+r^2+\beta^2 r^4\right)^2+4\zeta_1^2 r^2\left(1-\beta^2 r^2\right)^2} \\
&=\sqrt{\left(1-r^2\right)^2\left(1-\beta^2 r^2\right)^2+4\zeta_1^2 r^2\left(1-\beta^2 r^2\right)^2}=\sqrt{\left[\left(1-r^2\right)^2+4\zeta_1^2 r^2\right]\left(1-\beta^2 r^2\right)^2}\ .
\end{aligned}$$

Then Eq. (17) degrades to 1-DOF vibration transmissibility.

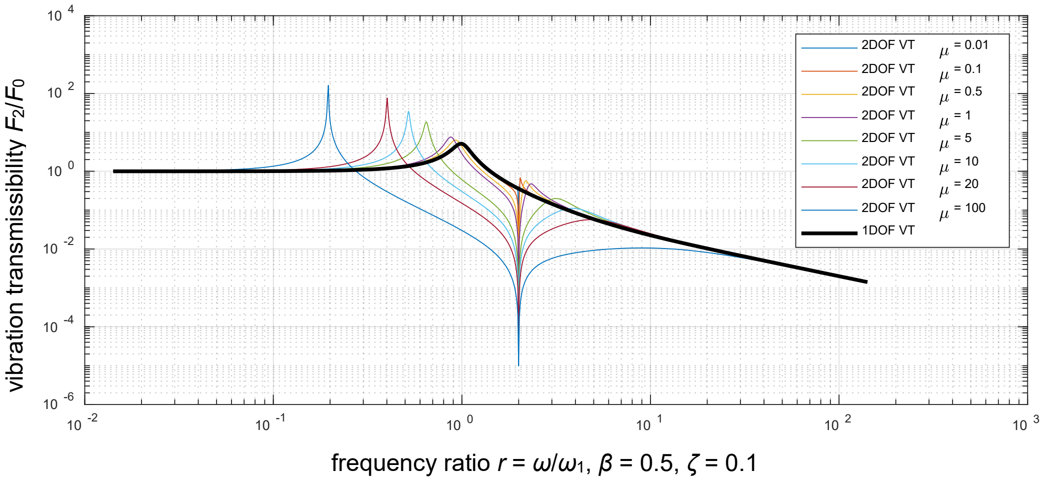

Figure 4. Effect of the mass ratio.

It is also observed in Figure 4 that the transmissibility curves approach the one-DOF curve as asymptote at high frequencies. Take the limit of Eq. (17) as $r \to \infty$:

$$\lim_{r\to\infty} \frac{\sqrt{\left(1-\beta^2 r^2\right)^2 \left(1+4\zeta_1^2 r^2\right)}}{\sqrt{\left\{1-\left[\left(1+\mu\right)\beta^2+1\right]r^2+\beta^2 r^4\right\}^2+4\zeta_1^2 r^2\left[1-\left(1+\mu\right)\beta^2 r^2\right]^2}} = \frac{2\zeta_1}{r} . \tag{21}$$

It is exactly the same limit of Eq. (1) as $r \to \infty$. In another word, the transmissibility decreases at a rate of 20 dB per decade at high frequencies. This conclusion is quite different from the transmissibility to the base. Taking the same limit of Eq. (13) indicates that the vibration transmissibility from m_1 to base decreases at a rate of 40 dB per decade at high frequencies.

2.4. Case C: $\zeta_1 \neq 0$ and $\zeta_2 \neq 0$

This is the most general case where the damping of both the degrees of freedom is included. For example, BRBF (buckling restrained braced frame) with dashpot dampers can significantly increase the damping of the structure. Unfortunately, due to the page limitation of the Inter Noise paper, the detailed discussions are ignored.

3. FINITE ELEMENT SIMULATION

A finite element model was developed using ANSYS APDL. The spring-damper element (COMBIN14) and structural mass element (MASS21) are used to model the two-DOF system. The following values are specified using Real Constant sets: m_1 = 2 kg, m_2 = 1 kg, k_1 = 50 N/m, k_2 = 100 N/m, c_1 = 1 kg/s, c_2 = 0. Therefore, μ = 2, β = 0.5, and ζ_1 = 5% (the blue curve in Figure 2).

The analysis type is Harmonic (`ANTYPE, HARMIC`). The frequency range is from 0 to 10 Hz and the number of substeps is 100. Proper boundary conditions are specified such that only vertical motions are allowed. A vertical force is applied at m_1.

In postprocessing the transmitted force is extracted by defining a variable which is the k_1 element's force at the lower node. The FEM result is compared with the theoretical result in Figure 5. The two results look almost identical in the entire frequency range. Only two zoom-in comparisons show slight differences.

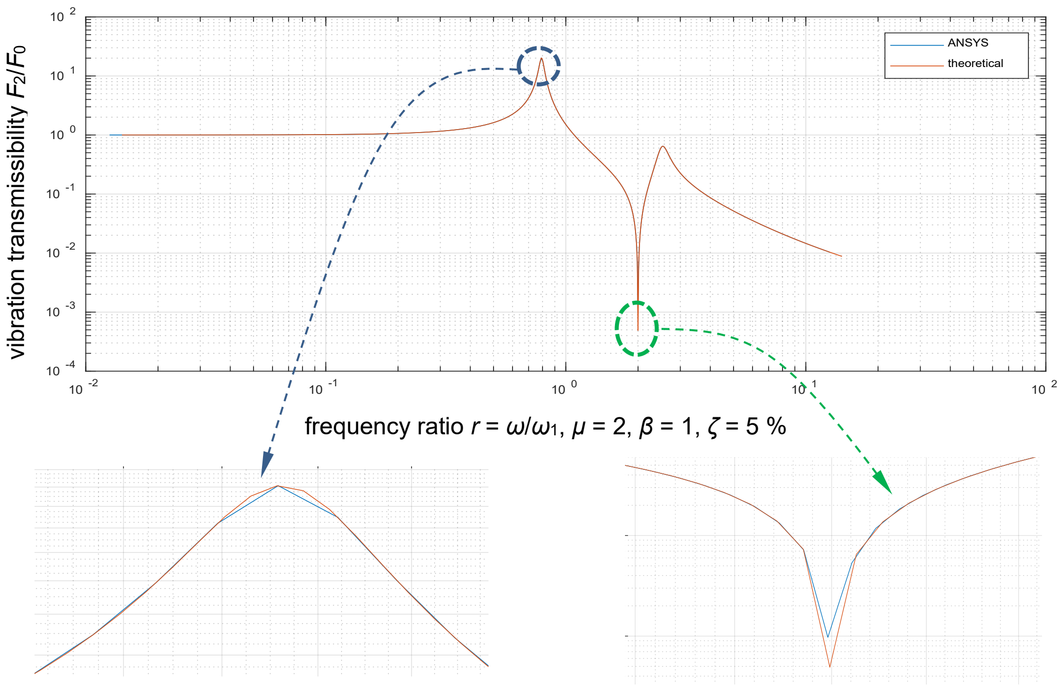

Figure 5. Comparison of transmissibility obtained using ANSYS and theoretical model.

More vibration transmissibility curves with various μ-β-ζ_1 combinations were created and compared with the theoretical results. They all agree with each other very well. Because of the limited page number, these plots are ignored in this paper.

4. CASE STUDY: DESIGN OF FLOATING FLOOR

This is a project of an office building. A fitness center was built on the level above four conference rooms. In order to reduce the impact sound transmission, a floating floor was constructed in the fitness center. The vibration transmissibility was also studied. The structural floor is 2-1/2" (63.5 mm) concrete slab on 3" (76.2 mm) steel deck. Steel beams are strengthened such that the structural floor's natural frequency is 6.7 Hz. The structural floor's weight is roughly 64 lb/ft^2 (312 kg/m^2). A 4" (101.6 mm) floating floor is supported by springs with a 2" (50.8 mm) air gap. By assuming dead load (52 lb/ft^2 or 254 kg/m^2) and live load (10 lb/ft^2 or 48.8 kg/m^2) of the fitness center, the floating floor's weight is roughly 62 lb/ft^2 (303 kg/m^2). Including the stiffness of entrapped air [10], the natural frequency of the floating floor is 15.6 Hz. So the mass ratio is almost unity $\mu \approx 1$, and the frequency ratio is $\beta = 2.3283$. Due to concrete and steel beamss the damping of the structural floor is negligible. So Eq. (17) was used to study the vibration transmitted from the floating floor to the structural floor.

Figure 6 illustrates the vibration transmissibility curves. It can be seen that the first resonance frequency is at 4.64 Hz. Also, the effect of damping is insignificant for frequencies lower than 10 Hz. These curves imply that spending more money to increase damping of the floating floor is not an effective way to reduce vibration.

The equipment in the fitness center is mainly elliptical trainers and weight lifting stations without dancing or aerobics activities. There is an additional layer of shock absorbent mat for the weight lifting area. So the exciting frequencies are typically lower than 3 Hz. The vibration transmissibility is less than 1.41 at frequencies lower than 3 Hz. In addition, the vibration level at frequencies close to the structural floor's natural frequency is significantly reduced. The only concern is that the second harmonic of some of the gym activities may be close to 4.64 Hz and got amplified.

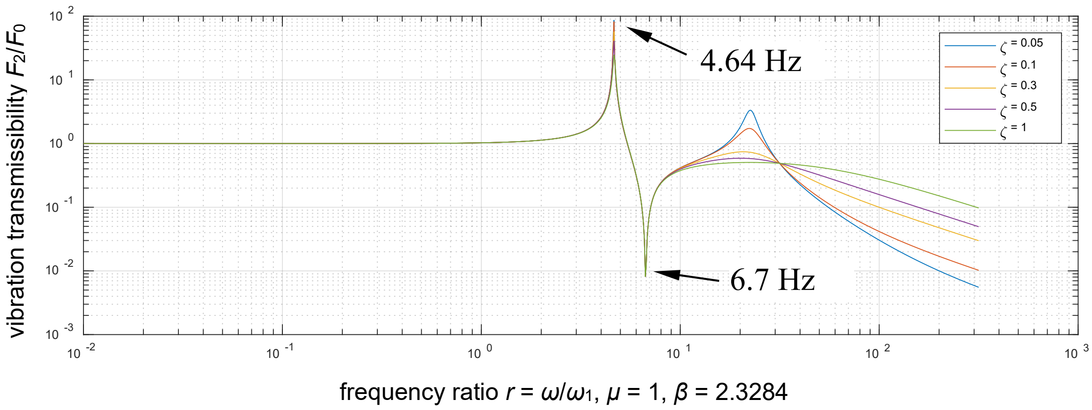

Figure 6. A floating floor design example.

5. CONCLUSIONS

The theoretical formulas of damped two-DOF vibration transmissibilities are derived for the first time in this paper. Different cases are discussed. Some details such as the effect of mass ratio, resonance frequencies, are deliberated. These theoretical formulas are validated by a finite-element model using ANSYS. A case study of floating floor design is also presented.

Other damping mechanisms, such as hysteresis damping will be investigated in future research.

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