Numerical study of high frequency nonlinear gas oscillation

Yano, Takeru[1]
Department of Mechanical Engineering, Osaka University
2-1, Yamada-oka, Suita, 565-0871, Japan

ABSTRACT

High frequency nonlinear gas oscillations in a one-dimensional finite space bounded by two flat plates are considered based on the numerical analysis of the Boltzmann–Krook–Welander equation (the BGK model of the Boltzmann equation), which is a nonlinear integro-differential equation for the velocity distribution function of gas molecules. The gas oscillation is assumed to be excited by the plates, which oscillate harmonically keeping the distance between them, like a gas motion in a vessel on a shaker. The initial and boundary value problem of the integro-differential equation is numerically solved with a finite difference and Simpson's rule. The solution at the resonance condition predicted from the continuum theory shows the shock wave in the steady oscillation state. The discontinuity in the velocity distribution function is also demonstrated.

1. INTRODUCTION

We consider the high frequency nonlinear gas oscillation in a one-dimensional finite space bounded by two flat plates. The plates oscillate harmonically in the normal direction, keeping their distance constant. When the distance and the frequency of oscillation satisfy a resonance condition, a resonant gas oscillation occurs and the amplitude of oscillation becomes large in time. If the nonlinear effect dominates the dissipation effect, the shock wave is formed in the oscillation process. The nonlinear gas oscillation in a closed tube with oscillating piston at one end has been studied by many authors [1–5]. The present problem, however, is different from them in that the gas oscillation is excited not by the oscillating

---

[1] yano@mech.eng.osaka-u.ac.jp

piston but by the oscillation of the gas-filled space bounded by two plates; this is a model of vibration of a vessel by a shaker. In particular, we are aiming to clarify the nonequilibrium effect on the gas oscillations, which appears in the case that the mean free path and mean free time of gas molecules cannot be neglected compared with the wavelength and the period of oscillation, respectively. Acoustics based on fluid mechanics (i.e., continuum theory) cannot describe such a situation. The kinetic theory of gases based on the Boltzmann equation (rarefied gas dynamics or molecular gas dynamics) is applied instead [6–8].

2. FORMULATION OF PROBLEM

Suppose that a finite space bounded by two flat plates with distance L is filled with a monatomic gas. At an initial state, the plates are at rest with temperature T_0, and the gas is in an equilibrium state of temperature T_0 and density ρ_0. The mean free path ℓ_0 and mean free time t_0 in the equilibrium state are estimated by [6–8]

$$\ell_0 = \frac{\mu_0}{\gamma_1 \rho_0 R T_0} \sqrt{\frac{8RT_0}{\pi}}, \quad t_0 = \ell_0 \sqrt{\frac{\pi}{8RT_0}} \tag{1}$$

where μ_0 is the viscosity coefficient of the gas in the equilibrium state, R is the specific gas constant, and γ_1 is a nondimensional constant depending on the model of molecular interaction in the gas. At time $t = 0$, the two plates begin oscillating harmonically with amplitude a and angular frequency ω. The temperature of the plates are kept at T_0 all the time. We numerically study the resulting gas motion on the basis of the kinetic theory of gases.

The problem is mathematically formulated with nondimensional variables defined as follows:

$$\hat{t} = \frac{t\sqrt{2RT_0}}{\ell_0}, \quad \hat{x}_i = \frac{x_i}{\ell_0}, \quad \zeta_i = \frac{\xi_i}{\sqrt{2RT_0}}, \quad \hat{f} = \frac{f}{\rho_0(2RT_0)^{3/2}} \tag{2}$$

where t is the time from the beginning of oscillation, x_i is the spacial coordinate ($i = 1, 2, 3$) (x_1 in the normal direction of the plate), ξ_i is the x_i-component of the molecular velocity, f is the velocity distribution function of the gas molecules. Note that, although the gas motion is assumed as one-dimensional in the physical space, the molecular velocity is the three-dimensional vector.

The governing equation is the Boltzmann–Krook–Welander equation (the BGK model of the Boltzmann equation), the nondimensional form of which can be written as [8]

$$\frac{\partial \hat{f}}{\partial \hat{t}} + \zeta_1 \frac{\partial \hat{f}}{\partial \hat{x}_1} = \frac{2\hat{\rho}}{\sqrt{\pi}} \left\{ \frac{\hat{\rho}}{(\pi\hat{T})^{3/2}} \exp\left[-\frac{(\zeta_j - \hat{u}_j)^2}{\hat{T}} \right] - \hat{f} \right\} \tag{3}$$

$$\begin{pmatrix} \hat{\rho} \\ \hat{\rho}\hat{u}_j \\ 3\hat{\rho}\hat{T} \end{pmatrix} = \int \begin{pmatrix} 1 \\ \zeta_j \\ 2(\zeta_j - \hat{u}_j)^2 \end{pmatrix} \hat{f} \, \mathrm{d}\zeta_1 \mathrm{d}\zeta_2 \mathrm{d}\zeta_3 \tag{4}$$

where $\hat{\rho} = \rho/\rho_0$, $\hat{u}_j = u_j/\sqrt{2RT_0}$ and $\hat{T} = T/T_0$ in Equation 3 are defined by the integrals of unknown function $\hat{f}$ over the whole velocity space in Equation 4. The

Boltzmann–Krook–Welander equation is therefore a nonlinear integro-differential equation.

The boundary condition on the left plate is

$$\hat{f} = \frac{\hat{\rho}_L}{(\pi)^{3/2}} \exp\left[-\left(\zeta_j - \delta_{1j}\frac{\mathrm{d}Z(\hat{t})}{\mathrm{d}\hat{t}}\right)^2\right] \quad \text{for } \zeta_1 > \frac{\mathrm{d}Z(\hat{t})}{\mathrm{d}\hat{t}} \quad \text{at } \hat{x}_1 = Z(\hat{t}) \qquad (5)$$

$$\hat{\rho}_L = -2\sqrt{\pi}\int_{\zeta_1 < \mathrm{d}Z(\hat{t})/\mathrm{d}\hat{t}}\left(\zeta_1 - \frac{\mathrm{d}Z(\hat{t})}{\mathrm{d}\hat{t}}\right)\hat{f}\bigg|_{\hat{x}_1 = Z(\hat{t})}\mathrm{d}\zeta_1\mathrm{d}\zeta_2\mathrm{d}\zeta_3 \qquad (6)$$

and that on the right plate is

$$\hat{f} = \frac{\hat{\rho}_R}{(\pi)^{3/2}} \exp\left[-\left(\zeta_j - \delta_{1j}\frac{\mathrm{d}Z(\hat{t})}{\mathrm{d}\hat{t}}\right)^2\right] \quad \text{for } \zeta_1 < \frac{\mathrm{d}Z(\hat{t})}{\mathrm{d}\hat{t}} \quad \text{at } \hat{x}_1 = Z(\hat{t}) + \hat{L} \qquad (7)$$

$$\hat{\rho}_R = 2\sqrt{\pi}\int_{\zeta_1 > \mathrm{d}Z(\hat{t})/\mathrm{d}\hat{t}}\left(\zeta_1 - \frac{\mathrm{d}Z(\hat{t})}{\mathrm{d}\hat{t}}\right)\hat{f}\bigg|_{\hat{x}_1 = Z(\hat{t})+\hat{L}}\mathrm{d}\zeta_1\mathrm{d}\zeta_2\mathrm{d}\zeta_3 \qquad (8)$$

where $Z(\hat{t})$ denotes the oscillation of the plates,

$$Z(\hat{t}) = \frac{M}{\Omega}\sqrt{\frac{5}{6}}(\cos\Omega\hat{t} - 1) \qquad (9)$$

The acoustic Mach number M, the nondimensional angular frequency Ω and the nondimensional distance between the plates $\hat{L}$ are defined as

$$M = \sqrt{\frac{6}{5}}\frac{a\omega}{\sqrt{2RT_0}}, \quad \Omega = \frac{\ell_0\omega}{\sqrt{2RT_0}}, \quad \hat{L} = \frac{L}{\ell_0} \qquad (10)$$

Equations 5–8 represent the diffuse-reflection condition [8]. The initial condition is

$$\hat{f} = (\pi)^{-3/2}\exp\left(-\zeta_j^2\right) \quad \text{at} \quad \hat{t} = 0 \quad \text{for} \quad 0 \leqq \hat{x}_1 \leqq \hat{L} \qquad (11)$$

In the case that $M \ll 1$, $\Omega \ll 1$ and $\hat{L} \gg 1$, the gas motion may be described by the classical acoustics. In the coordinate moving with the plate, $X = \hat{x} - Z(\hat{t})$, the governing equation can be written as

$$\frac{\partial^2 U}{\partial\hat{t}^2} - \frac{5}{6}\frac{\partial^2 U}{\partial X^2} = -\sqrt{\frac{5}{6}}M\Omega^2\sin\Omega\hat{t} \quad \text{for} \quad 0 \leqq X \leqq \hat{L} \qquad (12)$$

where $U = \hat{u}_1 - \mathrm{d}Z(\hat{t})/\mathrm{d}t$, and the boundary conditions are $U = 0$ at $X = 0$ and $U = 0$ at $X = \hat{L}$. . If we ignore the initial condition, the solution of Equation 12 is given as

$$U = \sqrt{\frac{5}{6}}M\sin\Omega\hat{t}\left[1 - \cos\left(\sqrt{\frac{6}{5}}\Omega X\right)\right]$$
$$+ \sqrt{\frac{5}{6}}M\tan\left(\sqrt{\frac{6}{5}}\frac{\Omega\hat{L}}{2}\right)\left[\sin\left(n\pi\sqrt{\frac{5}{6}}\frac{\hat{t}}{\hat{L}}\right)\sin\left(n\pi\frac{X}{\hat{L}}\right) - \sin\Omega\hat{t}\sin\left(\sqrt{\frac{6}{5}}\Omega X\right)\right] \qquad (13)$$

where n is an integer. Due to the singularity of tangent in Equation 13, we see that the resonance condition in the classical acoustics is

$$\sqrt{\frac{6}{5}}\Omega\hat{L} = n\pi \quad (n = 1, 3, 5, ...) \tag{14}$$

For example, for $n = 1$, we have

$$
\begin{aligned}
U &= \sqrt{\frac{5}{6}}M\sin\left(\sqrt{\frac{5}{6}}\frac{\pi\hat{t}}{\hat{L}}\right)\left[1 - \cos\left(\frac{\pi\hat{x}}{\hat{L}}\right)\right] \\
&+ \sqrt{\frac{10}{3}}\frac{M\hat{x}}{\hat{L}}\sin\left(\sqrt{\frac{5}{6}}\frac{\pi\hat{t}}{\hat{L}}\right)\cos\left(\frac{\pi\hat{x}}{\hat{L}}\right) + \frac{5M\hat{t}}{3\hat{L}}\cos\left(\sqrt{\frac{5}{6}}\frac{\pi\hat{t}}{\hat{L}}\right)\sin\left(\frac{\pi\hat{x}}{\hat{L}}\right) \\
&\approx \frac{5M\hat{t}}{3\hat{L}}\cos\left(\sqrt{\frac{5}{6}}\frac{\pi\hat{t}}{\hat{L}}\right)\sin\left(\frac{\pi\hat{x}}{\hat{L}}\right) \quad \text{for} \quad \hat{t} \gg 1 \tag{15}
\end{aligned}
$$

3. NUMERICAL METHOD

To solve the problem numerically, first of all, we eliminate two molecular velocity components ζ_2 ad ζ_3 [9]. Then, the velocity distribution function becomes a function of $\hat{x} = \hat{x}_1$, ζ_1 and $\hat{t}$. The left-hand side of Equation 3 is discretized with the second-order finite difference in space and the first-order finite difference in time. The integrals in the right-hand side of Equation 3 are reduced to integrals with respect to ζ_1, and evaluated by Simpson's rule.

4. NUMERICAL RESULT

In the following, we present typical examples of numerical results for the resonance of the fundamental mode, $\sqrt{(6/5)}\Omega\hat{L} = \pi$ ($n = 1$), and a non-resonance case, $\sqrt{(6/5)}\Omega\hat{L} = 2\pi$ ($n = 2$).

4.4.1. Velocity profiles

Figures 1–3 show the velocity profiles of every 1/8-cycle between 40th cycle and 50th cycle from the beginning of oscillation. In all cases shown here, the gas oscillations reach steady oscillation states. In a moderately low frequency case shown in Figure 1, we can see a shock wave in the resonance mode and the waveform distortion due to the nonlinearity in the non-resonance mode. A shock wave in the resonance mode and the waveform distortion in the non-resonance mode also take place in a high frequency case shown in Figure 2. In Figure 3, however, the shock wave disappears because the nonequilibrium effect works dissipatively.

4.4.2. Velocity distribution function

In Figure 4, we present the velocity distribution function of the gas molecules on the left plate ($X = 0$) at every 1/8-cycle in the 50th cycle. The dashed lines correspond to the velocity of plate and the velocity distribution functions are discontinuous at those velocities.
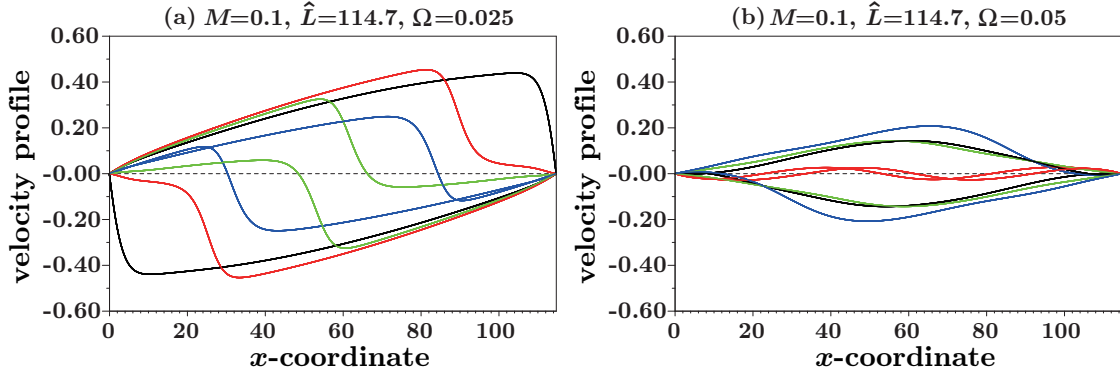
Figure 1: The velocity profiles in a moderately low frequency case. (a): The fundamental mode ($n = 1$). (b): A non-resonance mode ($n = 2$).
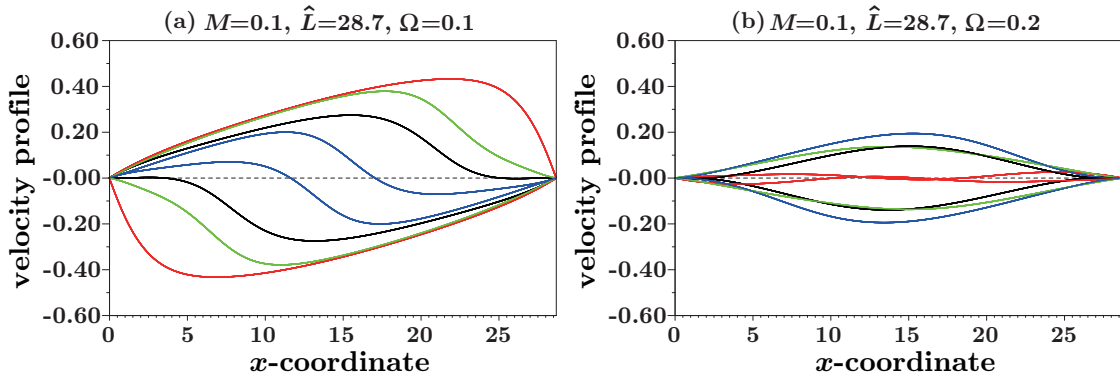

Figure 2: The velocity profiles in a high frequency case. (a): The fundamental mode ($n = 1$). (b): A non-resonance mode ($n = 2$).
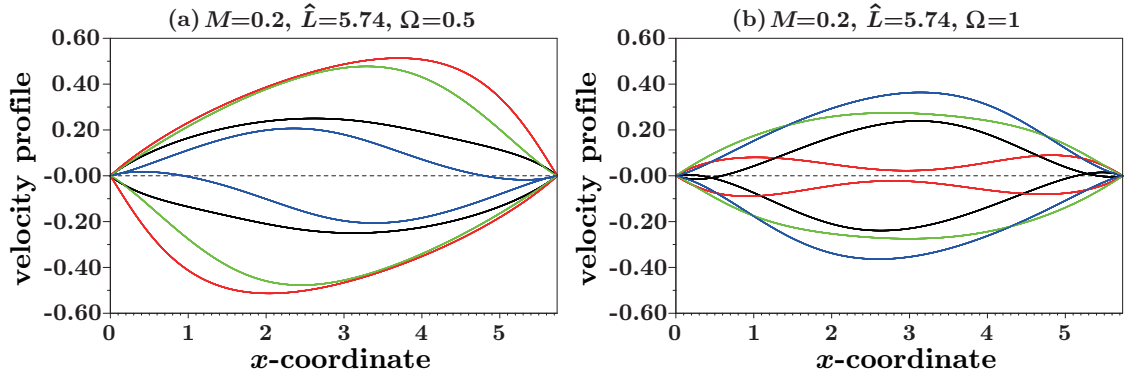

Figure 3: The velocity profiles in a very high frequency case. (a): The fundamental mode ($n = 1$). (b): A non-resonance mode ($n = 2$).

5. CONCLUSIONS

We have presented the numerical results of high frequency gas oscillation obtained from the numerical solutions of the Boltzmann–Krook–Welander equation and the diffuse-reflection condition on the oscillating plates. As a result, it is confirmed that the nonequilibrium effect works dissipatively. The discontinuities in the velocity distribution function are also demonstrated. The results in a moderately low frequency case qualitatively agree with those predicted by the continuum theory.
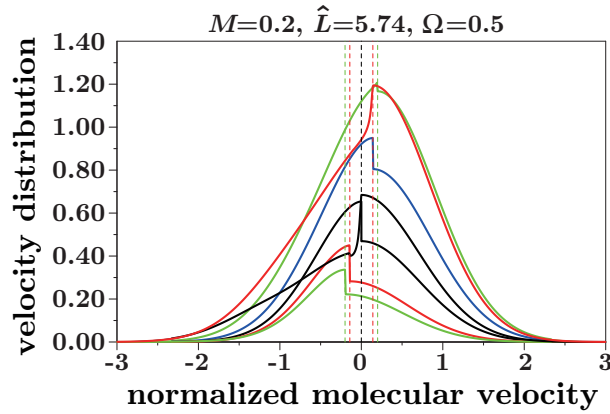
Figure 4: The velocity distribution function of gas molecules on the plate in a very high frequency case. The fundamental mode ($n = 1$). The dashed lines correspond to the velocity of plate.

However, as can be seen from the fact that the velocity distribution function is not a Maxwellian, the analysis of high frequency gas oscillation inevitably requires the kinetic theory of gases.

6.   REFERENCES

[1] W. Chester. Resonant oscillations in closed tubes. *J. Fluid Mech.*, 18:44–66, 1964.

[2] B. R. Seymour and M. P. Mortell. Resonant acoustic oscillations with damping: small rate theory. *J. Fluid Mech.*, 58:353–373, 1973.

[3] N. Rott. The heating effect connected with non-linear oscillations in a resonance tube. *Z. Angew. Math. Phys.*, 25:619–634, 1974.

[4] A. Goldshtein, P. Vainshtein, M. Fichman, and C. Gutfinger. Resonant gas oscillations in closed tubes. *J. Fluid Mech.*, 322:147–163, 1996.

[5] K. Aoki, S. Kosuge, T. Fujiwara, and T. Goudon. Unsteady motion of a slightly rarefied gas caused by a plate oscillating in its normal direction. *Phy. Rev. Fluids*, 2:013402, 2017.

[6] C. Cercignani. *The Boltzmann Equation and Its Applications.* Springer, 1988.

[7] M. N. Kogan. *Rarefied Gas Dynamics.* Plenum, 1969.

[8] Y. Sone. *Molecular Gas Dynamics.* Birkhäuser, 2007.

[9] C. K. Chu. Kinetic-theoretic description of the formation of a shock wave. *Phys. Fluids*, 8:12–22, 1965.