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# Exact solution of Euler-Bernoulli equation for acoustic black holes via generalized hypergeometric differential equation

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**ABSTRACT**

**Acoustic black hole (ABH), a thin wedge type structure with its thickness is tailored according to the power-law of power (m) greater than or equal to two, has received much attention from the researchers due to its potential as a light and effective absorber of flexural waves propagating in beams or plates. In this paper, the Euler-Bernoulli equation for the ABH of m>2 is reformulated into the form of a generalized hypergeometric differential equation. The exact solution is then derived in terms of generalized hypergeometric functions (pFq) where p=0 and q=3 by classifying the power m into four cases. The derived solution is in linearly independent form without singularities for arbitrary m. In addition, by using the exact solution, the displacement field of a uniform beam with an ABH and the reflection coefficient from the ABH are calculated to show the applicability of the present solution. This paper aims at establishing a mathematical and theoretical foundation for the study of the ABHs.**

**Keywords:** Acoustic black hole, Elastic wave, Euler-Bernoulli beam, Wave absorption
**I-INCE Classification of Subject Number: 47**

## 1. INTRODUCTION

The acoustic black hole (ABH) effect has received much attention from researchers during the last decade as a potential technique to achieve light and effective absorption of flexural waves in thin structures such as beams or plates. The ABH is a thin wedge whose thickness is reduced according to the power-law function of power ($m$) greater than or equal to two [1]. In theory, when the thickness of the ABH gradually decreases to zero, flexural waves that propagate to the tip of the ABH are slowed down to zero-speed and dissipated completely during their travel. However, since the tip of the ABH is always truncated and has a finite thickness in reality, the waves are unavoidably reflected from the tip [1]. To absorb the waves near the truncated tip, a method of treating the tip of the ABH with a viscoelastic damping layer was proposed [2].

Following the early researches, numerous studies have been conducted in recent years to study the ABH effect further and to apply the ABH in practice. Examples of experimental studies are listed as follows: the experimental realization of the ABH [3], harvesting the energy within the tip region of the ABH [4,5], and measurement of the

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sound radiated from the vibration of the ABH [6]. Examples of computational studies are listed as follows: the study [7] on vibroacoustic performance of the two-dimensional ABH, and the geometrical modification [8] on the ABH with a spiral shaped-baseline. In addition to the diverse computational and experimental studies on the ABH, there have been several theoretical studies on the flexural wave motion in the ABH and the vibration of the ABH-attached structures. O'Boy et al. [9] investigated the vibration of a rectangular plate attached to an ABH by solving the eikonal equation for the ABH. Georgiev et al. [10] obtained the reflection matrices of an ABH by numerically solving a matrix Riccati equation. Denis et al. [11] presented a multimodal model of an ABH to study the effect of tip imperfections on the reflection characteristics of the ABH. Aklouche et al. [12] investigated the flexural wave scattering from a two-dimensional ABH embedded in an infinite plate by deriving the exact solution for the two-dimensional ABH of $m = 2$. Tang et al. [13] established a semi-analytic model that uses Mexican hat wavelets as basis functions to approximate the displacement field of a uniform beam attached to an ABH.

In this paper, the Euler-Bernoulli equation for the ABH is studied mathematically, and the exact solution is derived for arbitrary $m$ ($\geq 2$). In Section 2.1, it is shown that the original form of the Euler-Bernoulli equation for $m = 2$ belongs to the Cauchy-Euler equation and the solution is obtained as a linear combination of monomials. In Section 2.2, the Euler-Bernoulli equation for $m > 2$ is reformulated into the generalized hypergeometric differential equation by changing the variables. Then, four linearly independent solutions of the Euler-Bernoulli equation for $m > 2$ are obtained in regular form without singularities by dividing the power $m$ into four cases.

## 2. Exact solution for the ABH in the case of arbitrary power m

The governing equation for a beam neglecting the rotational inertia and shear deformation, i.e., an Euler-Bernoulli beam, is written as the following for the time-harmonic transverse displacement $W(x,t)$:

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 W}{\partial x^2}\right)+\rho A\frac{\partial^2 W}{\partial t^2}=0. \tag{1}$$

Here, $x$ is the distance along the neutral axis of the beam and $t$ is the time. $\rho$ and $E$ are the mass density and Young's modulus, and $A$ and $I$ are the cross-sectional area and second moment of area, respectively. Assuming the time dependence of $W$ to be of the form $\mathrm{e}^{\mathrm{j}\omega t}$ ($\mathrm{j}=\sqrt{-1}$) with angular frequency $\omega$, Equation 1 can be rewritten as

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)-\rho A\omega^2 w(x)=0 \tag{2}$$

in terms of the time-harmonic transverse displacement $w(x)$.

When the cross-section is rectangular and the thickness is tailored in the form of the power-law function ($h(x)=\varepsilon x^m$ for $m>0, x>0$) as depicted in Fig. 1, Equation 2 is recast into Equation 3:

$$x^{2m}\frac{\mathrm{d}^4 w}{\mathrm{d}x^4}+6mx^{2m-1}\frac{\mathrm{d}^3 w}{\mathrm{d}x^3}+3m(3m-1)x^{2m-2}\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}-\xi w(x)=0, \tag{3}$$

where $\xi$ is a constant $12\rho\omega^2/E\varepsilon^2$ depending on the frequency, material properties and shape of the beam.

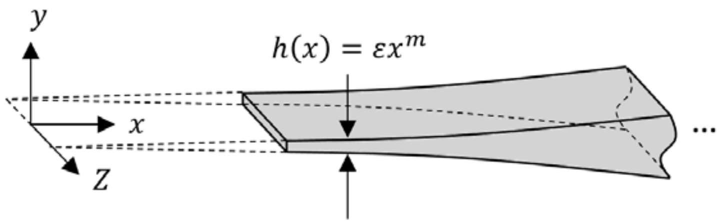


*Figure 1: A schematic of an Euler-Bernoulli beam whose thickness is tailored in the form of the power-law function ($h(x) = \varepsilon x^m$ for $m > 0, x > 0$) with a constant width.*

In case that the power *m* is equal to or greater than 2, the tailored beam is referred to as the ABH [2], and thus, the exact solution of the Euler-Bernoulli equation for the ABH can be obtained by solving Equation 3.

In Section 2.1, the exact solution for the ABH of *m* = 2 is derived, and in Section 2.2, the exact solution for the ABH of *m* > 2 is derived. In Section 2.3, by using the present exact solutions for the ABH, two problems related to ABH are solved and compared with a numerical simulation and an existing theory based on geometrical acoustics.

**2.1 Exact solution for the ABH of m = 2**

In the case of *m* = 2, Equation 3 is rewritten as Equation 4.

$$x^4 w''''(x) + 12x^3 w'''(x) + 30x^2 w''(x) - \xi w(x) = 0 \tag{4}$$

Since Equation 4 belongs to the Cauchy-Euler equation, the solution can be expressed by a linear combination of monomials as obtained in Equation 5

$$w(x) = x^{-\frac{3}{2}} \left\{ C_1 x^{-\frac{1}{2}\sqrt{17-4\sqrt{4+\xi}}} + C_2 x^{-\frac{1}{2}\sqrt{17+4\sqrt{4+\xi}}} + C_3 x^{\frac{1}{2}\sqrt{17-4\sqrt{4+\xi}}} + C_4 x^{\frac{1}{2}\sqrt{17+4\sqrt{4+\xi}}} \right\}, \tag{5}$$

where $C_1$, $C_2$, $C_3$ and $C_4$ are constants determined by the boundary conditions.

**2.2 Exact solution for the ABH of m > 2**

In the case of *m* > 2, define the variable *z* as follows:

$$z = \frac{\xi}{(4-2m)^4} x^{4-2m}. \tag{6}$$

By using a differential operator $\vartheta \left(= z\frac{\mathrm{d}}{\mathrm{d}z}\right)$, Equation 3 can be recast into the following:

$$\left\{\vartheta\left(\vartheta + \frac{3-2m}{4-2m} - 1\right)\left(\vartheta + \frac{2+m}{4-2m} - 1\right)\left(\vartheta + \frac{1+m}{4-2m} - 1\right) - z\right\} w(z) = 0. \tag{7}$$

Equation 7 belongs to a type of special ordinary differential equation named the generalized hypergeometric differential equation (GHGE) [14]. The general form of the GHGE is written as

$$\{\vartheta(\vartheta + \alpha_1 - 1)(\vartheta + \alpha_2 - 1) \ldots (\vartheta + \alpha_q - 1) - z(\vartheta + \gamma_1)(\vartheta + \gamma_2) \ldots (\vartheta + \gamma_p)\} w(z) = 0, \tag{8}$$

where $\alpha_1, \alpha_2, \dots, \alpha_q$ are referred to as denominator-parameters and $\gamma_1, \gamma_2, \dots, \gamma_p$ are referred to as numerator-parameters.

Exact solution of the GHGE can be obtained in terms of a special function called the generalized hypergeometric function (GHF) [14] which is defined as

$$ {}_p\mathrm{F}_q(\gamma_1, \gamma_2, \dots, \gamma_p; \alpha_1, \alpha_2, \dots, \alpha_q; z) = \sum_{n=0}^{\infty} \frac{(\gamma_1)_n (\gamma_2)_n \cdots (\gamma_p)_n}{(\alpha_1)_n (\alpha_2)_n \cdots (\alpha_q)_n} \frac{z^n}{n!}. \tag{9}$$

Here, $(\cdot)_n$ denotes the Pochhammer symbol or the rising factorial defined as below:

$$(a)_n = \begin{cases} \prod_{k=0}^{n-1} (a+k), & n \in \mathbb{N}, \\ 1, & n = 0, \end{cases} \tag{10}$$

where $\mathbb{N}$ denotes the set of all natural numbers.

By comparing Equation 7 with Equation 8, it can be seen that Equation 7 belongs to the GHGE in the case of $p$=0 and $q$=3, and the corresponding denominator-parameters are $\alpha_1 = \frac{3-2m}{4-2m}$, $\alpha_2 = \frac{2+m}{4-2m}$ and $\alpha_3 = \frac{1+m}{4-2m}$.

### 2.3 Explicit form of exact solution

Let $\alpha_0 = 1$, $\alpha_1 = \frac{3-2m}{4-2m}$, $\alpha_2 = \frac{2+m}{4-2m}$ and $\alpha_3 = \frac{1+m}{4-2m}$, and let $\beta_{ij} = 1 - \alpha_i + \alpha_j$ for $i, j = 0,1,2,3$. Then, we have the following 1-tensor and 2-tensor:

$$\boldsymbol{\alpha} = (\alpha_0 \quad \alpha_1 \quad \alpha_2 \quad \alpha_3), \tag{11}$$

$$\boldsymbol{\beta} = \begin{pmatrix} \beta_{00} & \beta_{01} & \beta_{02} & \beta_{03} \\ \beta_{10} & \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{20} & \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{30} & \beta_{31} & \beta_{32} & \beta_{33} \end{pmatrix} = \begin{pmatrix} 1 & 1-\alpha_0+\alpha_1 & 1-\alpha_0+\alpha_2 & 1-\alpha_0+\alpha_3 \\ 1-\alpha_1+\alpha_0 & 1 & 1-\alpha_1+\alpha_2 & 1-\alpha_1+\alpha_3 \\ 1-\alpha_2+\alpha_0 & 1-\alpha_2+\alpha_1 & 1 & 1-\alpha_2+\alpha_3 \\ 1-\alpha_3+\alpha_0 & 1-\alpha_3+\alpha_1 & 1-\alpha_3+\alpha_2 & 1 \end{pmatrix}. \tag{12}$$

By using the tensors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ defined in the previous section, the general solution of Equation 7, $w(x)$, can be expressed by a linear combination of the four independent functions in terms of the generalized hypergeometric series of ${}_0\mathrm{F}_3$: (See the reference [15] for detailed procedures)

$$\begin{gathered} w(x) = C_0 w_0(x) + C_1 w_1(x) + C_2 w_2(x) + C_3 w_3(x) \text{ where} \\ w_0(x) = {}_0\mathrm{F}_3(-; \beta_{01}, \beta_{02}, \beta_{03}; z), \\ w_1(x) = z^{\frac{1}{4-2m}} \, {}_0\mathrm{F}_3(-; \beta_{10}, \beta_{12}, \beta_{13}; z), \\ w_2(x) = z^{\frac{2-3m}{4-2m}} \, {}_0\mathrm{F}_3(-; \beta_{20}, \beta_{21}, \beta_{23}; z), \\ w_3(x) = z^{\frac{3-3m}{4-2m}} \, {}_0\mathrm{F}_3(-; \beta_{30}, \beta_{31}, \beta_{32}; z). \end{gathered} \tag{13}$$

## 3. CONCLUSIONS

In this study, we provided a rigorous foundation for theoretical and mathematical research for the ABHs by deriving the exact solution of the Euler-Bernoulli equation for ABH in the cases of arbitrary power $m$ ($\geq 2$). For $m = 2$, the original form of the Euler-Bernoulli equation belongs to the Cauchy-Euler equation, and the exact solution was obtained as a linear combination of four monomials. For $m > 2$, the Euler-Bernoulli equation was transformed into the generalized hypergeometric differential equation via two steps of change of variables, and the exact solution was derived in terms of four generalized hypergeometric series.

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