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Flexural-Torsional Dynamics of Locally Resonant Finite Metamaterial

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ABSTRACT

This paper deals with the flexural-torsional dynamics of finite beams with many spring-mass absorbers periodically attached. Especial emphasis is given to the description of the vibration absorption properties and the bandgap formation. As an example of structures presenting dynamic coupling between bending and torsion, a channel beam is considered. Vlasov´s theory is employed to formulate mathematically the problem. An exact analytical solution is given for the free and forced vibration of the metamaterial beam. The attenuation of the dynamic forced response of the beam, by selecting appropriately the absorbers' parameters, is numerically analysed with the help of the obtained analytical solution.
Natural frequencies of the system are used for establishing a criterion to determine the bandgap edge. Finally, an approximate simple procedure is proposed to estimate the bandgap frequency range using only the inertial properties of the beam. This procedure is applicable to arbitrary end conditions.

Keywords: bandgap, Vlasov beams, locally resonant, metamaterials, vibration attenuation.
I-INCE Classification of Subject Number: 42, 46.

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1. INTRODUCTION

In the last years, the study of metamaterials consisting of locally resonant periodic structures has attracted the attention of several researchers due to their interesting properties related to sound and vibration attenuation in low frequencies. In fact, metamaterial structures exhibit frequency ranges, known as bandgaps, where elastic waves cannot propagate. This kind of structures finds interesting potential applications as broadband vibration absorbers, wave guiding and filtering.

A simple and non-expensive way to construct a metamaterial is by coupling, in a periodic arrangement, small spring-mass subsystems to beams or plates.

The idea of using locally resonant acoustic metamaterials is borrowed from the theory of electromagnetic waves that present bandgaps at wavelengths much longer that the distance between atoms. Liu et al. [1] have demonstrated that locally resonant acoustic metamaterials display bandgaps similar to electromagnetic materials. Yu et al. [2] have analysed the formation of flexural bandgaps in Euler-Bernoulli beams coupled to spring-mass subsystems with two-degrees of freedom. Sun et al. [3] present a theoretical development and numerical validation of metamaterial beams, with many small spring-mass-damper subsystems integrated at locations along the length, for broadband absorption of transverse elastic waves. Recently, Cveticanin and Mester [4] presented an overview of the theory of metamaterial beams.

In general, the above mentioned works, and other related, have analysed the bandgap formation based on the features of waves travelling along infinite structures made from repeated unit cells. This approach is appropriate for studying absorption properties in very long structures. However, for finite structures, especially in the range of low frequencies, it is more convenient to use methods of modal analysis. In this connection, a new idea was recently proposed by Sugino et al. [5] in order to determine the bandgap behaviour of finite length Bernoulli-Euler beams with periodically attached spring-mass resonators. Their proposal consists in considering the modal analysis of a finite beam with an infinite number of resonators as the dual problem of wave propagation through an infinite periodic beam. This new concept allows to consider in a unified way the modal behaviour of finite beams and the dispersion properties of infinite beams. With the developed methodology Sugino et al. [6] have obtained a simple formula for placing a bandgap in a desired frequency range. This is very useful for design purposes.

Sugino’s approach was firstly developed for flexural vibrations in beams and later extended to consider, in a unified form, uncoupled longitudinal or torsional vibrations of rods and transverse vibrations in plates with many coupled absorbers. However, the important case of coupled flexural-torsional vibrations of beams having periodically attached spring-mass resonators was not studied. This kind of structures is very used in several applications of engineering systems. One of the few works analysing flexural-torsional vibration bandgaps in infinite beams made of periodic material was developed by Jian-Yu et al. [7].

In this work, the attenuation of the vibrations of finite thin-walled beams by using many small spring-mass absorbers is analysed. As an example of structures presenting dynamic coupling between bending and torsion, a channel beam is considered. Particular attention is given to the mechanism of bandgap formation. Following Sugino’s approach a modal analysis for finite thin-walled beams with an infinite of vibration absorbers is employed. Vlasov´s theory [8, 9] is employed to formulate mathematically the problem. An exact analytical solution is given for the free and forced vibration of the metamaterial beam. The attenuation of the dynamic forced response of the beam, by selecting appropriately the absorbers parameters, is numerically analysed with the help of the obtained analytical solution. Natural frequencies of the system are used for establishing a criterion to determine the bandgap edge. Finally, an approximate simple procedure is proposed to estimate the bandgap frequency range using only the inertial properties of the beam. This procedure is applicable to arbitrary end conditions.

2. FLEXURAL-TORSIONAL DYNAMICS OF THIN-WALLED METAMATERIAL BEAMS

2.1 Governing equations

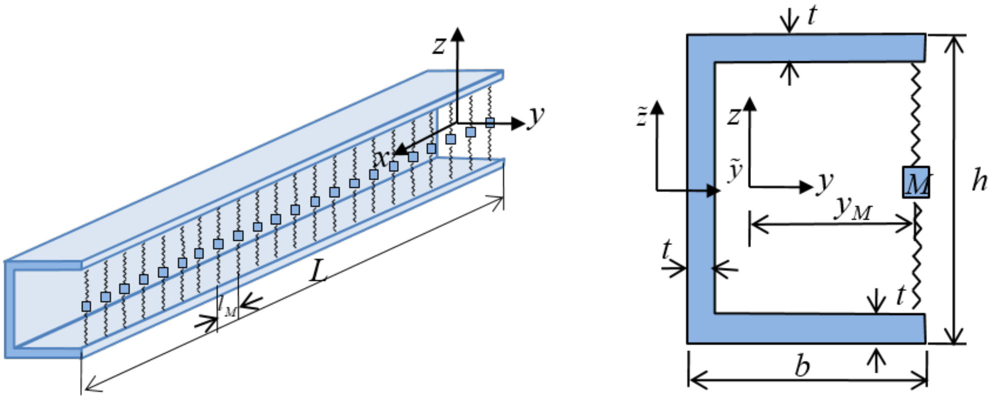

Figure 1: Thin-walled beam with spring-mass resonators periodically attached

The thin-walled beam with N attached spring-mass absorbers shown in Figure 1 is considered. Taking into account that the cross sectional shape has only one symmetry axis, the beam will present coupled flexural-torsional dynamics. Such kind of structure may be described mathematically according to the Vlasov theory. The corresponding governing equations are given by [7, 8]:

$$
\begin{aligned}
&-\frac{\partial^2 M}{\partial x^2}+\rho A\frac{\partial^2 w_s}{\partial t^2}-\rho A y_s\frac{\partial^2 \phi}{\partial t^2}=\sum_i K\left(\xi_i-\left(w_{si}+\phi_i y_M\right)\right)\delta_i+q_z\cos(\omega t),\\
&\frac{\partial^2 B}{\partial x^2}-\frac{\partial T_{sv}}{\partial x}-\rho A y_s\frac{\partial^2 w_s}{\partial t^2}+\rho I_s\frac{\partial^2 \phi}{\partial t^2}=\sum_i K y_M\left(\xi_i-\left(w_{si}+\phi_i y_M\right)\right)\delta_i+m_t\cos(\omega t),\qquad (1)\\
&M\frac{d^2\xi_i}{dt^2}+K\xi_i=K\left(w_{si}+\phi_i y_M\right),\qquad i=1,\ldots,N
\end{aligned}
$$

where w_s and ϕ are the transverse displacement (corresponding to the shear center) and the torsional rotation, ξ_i is the displacement of the mass located at x_i, ρ, A, I_s, y_s are the material density, cross-sectional area, cross-sectional moment of inertia with respect to the shear center and shear center coordinate with respect to the centroid, respectively ; K and M are the stiffness and the mass of each absorber , y_M is the cross-sectional coordinate of the absorber location respect to the centroid, q_z, m_t, ω correspond to transverse and twisting distributed loadings and external frequency respectively, N is the number of resonators and $\delta_i=\delta(x-x_i)$ is the Dirac function.

On the other hand, M_z, T_{sv} and B are the bending moment, Saint-Venant twisting moment and bi-moment respectively. Their relationships with the displacements are given by:

$$M_z = -EI\frac{\partial^2 w_s}{\partial x^2},\ \ T_{sv} = GJ\frac{\partial \phi}{\partial x},\ \ B = EC_w\frac{\partial^2 \phi}{\partial x^2}, \tag{2}$$

where E, G, J, C_w are the elasticity modulus, transverse elasticity modulus, torsional constant and warping constant. Obviously, the System 1 has appropriate boundary conditions associated.

In order to solve the System 1, harmonic vibrations are assumed:

$$w_s = W_s(x)\cos(\omega t),\ \phi = \psi(x)\cos(\omega t),\ \xi_i = \overline{\xi_i}\cos(\omega t) \tag{3}$$

Substituting Expressions 3 into Equations 1 and 2 and eliminating $\overline{\xi_i}$ of the resultant system, the following equations are obtained:

$$\begin{aligned} &EI\frac{\partial^4 W_s}{\partial x^4} - \overline{\rho A}\omega^2 W_s + \overline{\rho A y_s}\omega^2 \psi = q_z(x) \\ &EC_w\frac{\partial^4 \psi}{\partial x^4} - GJ\frac{\partial^2 \psi}{\partial x^2} + \overline{\rho A y_s}\omega^2 W_s - \overline{\rho I_s}\omega^2 \psi = m_t(x) \end{aligned} \tag{4}$$

where $\overline{\rho A}$, $\overline{\rho A y_s}$ and $\overline{\rho I_s}$ are generalized sectional inertia properties expressed as:

$$\begin{aligned} &\overline{\rho A} = \rho A\left(1 + \frac{\alpha}{1-(\omega/\omega_M)^2}\right),\ \ \overline{\rho A y_s} = \rho A y_s\left(1 - \alpha\frac{y_M}{y_s}\frac{1}{1-(\omega/\omega_M)^2}\right) \\ &\overline{\rho I_s} = \rho I_s\left(1 + \alpha\frac{A y_M^2}{I_s}\frac{1}{1-(\omega/\omega_M)^2}\right) \end{aligned} \tag{5}$$

with

$$\omega_M = \sqrt{\frac{k}{M}} \qquad \alpha = \frac{M_{total}}{\rho A L} \qquad M_{total} = NM \tag{6}$$

ω_M is the natural frequency of an isolated absorber and α is the ratio between the total mass of the attached resonators and the beam mass. It has to be noted that for obtaining Equations 4 and 5 an infinite absorbers approximation was employed. That is to say Dirac functions were substituted by $1/l_M$ where l_M is the distance between absorbers. In fact, absorbers were considered to be continuously distributed. This approximation is consistent for many absorbers placed regularly along the beam [5]. On the other hand, the following relationships have been considered:

$$\frac{K}{l_M} = \frac{K}{M}\frac{M}{l_M}\frac{N}{N} = \omega_M^2\frac{M_{total}}{L} = \omega_M^2\alpha\rho A \tag{7}$$

2.2 Free and forced vibrations of simply supported thin-walled metamaterial beams

Consider the beam with simply supported ends mathematically expressed as:

$$w_s = M_z = B = \phi = 0,\ x = 0,\ x = L \qquad \text{at } x = 0,\ x = L \tag{8}$$

The exact solution for the Systems 4-7 may be obtained by expressing the unknowns and the loadings as Fourier series:

$$W_s = \sum_n \alpha_n sin\frac{n\pi x}{L}, \quad \psi = \sum_n \beta_n sin\frac{n\pi x}{L}, \tag{9}$$

$$q_z = \sum_n q_n sin\frac{n\pi x}{L},\ m_t = \sum_n m_n sin\frac{n\pi x}{L}. \tag{10}$$

It has to be noted that with Expressions 9, the Boundary Conditions 8 are identically satisfied. Now, substituting Expressions 9 and 10 into Equations 4-7, one can obtain the following two-equation algebraic system for every value of *n*.

$$\begin{aligned} &\alpha_n\left(\frac{EI\pi^4}{\lambda_n^4} - \overline{\rho A\omega^2}\right) + \beta_n\left(\overline{\rho A y_s\omega^2}\right) = q_n \\ &\alpha_n\left(\overline{\rho A y_s\omega^2}\right) + \beta_n\left(\frac{EC_w\pi^4}{\lambda_n^4} + \frac{GJ\pi^2}{\lambda_n^2} - \overline{\rho I_s\omega^2}\right) = m_n \end{aligned} \tag{11}$$

The solution of the above algebraic linear system may be expressed in the following form:

$$\begin{aligned} \alpha_n &= \frac{q_n\left(\frac{EC_w\pi^4}{\lambda_n^4} + \frac{GJ\pi^2}{\lambda_n^2} - \overline{\rho I_s\omega^2}\right) - m_n\overline{\rho A y_s\omega^2}}{\Delta_n} \\ \beta_n &= \frac{-q_n\overline{\rho A y_s\omega^2} + m_n\left(\frac{EC_w\pi^4}{\lambda_n^4} + \frac{GJ\pi^2}{\lambda_n^2} - \overline{\rho I_s\omega^2}\right)\left(\frac{EI\pi^4}{\lambda_n^4} - \overline{\rho A\omega^2}\right)}{\Delta_n} \end{aligned} \tag{12}$$

where:

$$\Delta_n = \left(\frac{EI\pi^4}{\lambda_n^4} - \overline{\rho A\omega^2}\right)\left(\frac{EC_w\pi^4}{\lambda_n^4} + \frac{GJ\pi^2}{\lambda_n^2} - \overline{\rho I_s\omega^2}\right) - \left(\overline{\rho A y_s\omega^2}\right)^2, \quad \lambda_n = L/n \tag{13}$$

n=1, 2 ,3,…

To determine the natural frequencies of free vibration ($q_z = m_t = 0$), the determinant of the system should be null for every value of *n:*

$$\Delta_n(\omega_n) = 0, \qquad n = 1,2,3,... \tag{14}$$

For the structure without absorbers, Expression 14 constitutes a biquadratic equation whose roots are two natural frequencies for every value of *n*.

$$\alpha = 0 \rightarrow \omega_{n1}^{0},\ \omega_{n2}^{0} \tag{15}$$

For the case of the beam with attached absorbers, Expression 14 is a bicubic polynomial equation having three roots. It is possible to demonstrate that two of them ω_{nL} and ω_{nU} are limiting ω_M from below and above:

$$\alpha \neq 0 \rightarrow \omega_{nL}, \omega_M, \omega_{nU} \tag{16}$$

3. ATTENUATION OF DYNAMIC RESPONSE AND BANDGAP FORMATION

3.1 Analysis of forced response

The practical use of this kind of metamaterial beams is to attenuate amplitudes of vibration in comparison with the original structure (without resonators). This analysis may be performed in a quick manner by calculating the deformed shape of the beam with or without absorbers (α=0) using Expressions 12 and 9, and then comparing the vibration amplitudes. In order to obtain vibration absorption near certain target frequency ω_T (generally a certain natural frequency of the system without absorbers), the resonator frequency ω_M should be selected slightly lower (for example $\omega_M = 0.9\omega_T$). In fact, according to Equation 16, the resonance of the original system (associated with ω_T) will be transformed into two resonances, one of each side of the target frequency (at ω_{nL} and ω_{nU}). Thus, in the frequency range (ω_{nL}, ω_{nU}) the vibration amplitudes could be attenuated. For example, Figure 2 presents (in a semi-logarithmic scale) the amplitudes of transverse vibrations (corresponding to *n*=3) at the middle of the beam without (α=0) and with absorbers (α=0.5) for the numerical data given in the Appendix. The local resonance ω_M has been selected as $0.9\,\omega_T$, being the target frequency coincident with one of the natural frequencies of the original system (3942 *rad/seg* for *n*=3). As shown, the absorbers caused the elimination of the resonance corresponding the original system although two new lateral resonances have emerged, one on each side of the target frequency ($\omega_{nL} = 3150$ and $\omega_{nU} = 5000$ *rad/seg*). The dynamic response was attenuated approximately in the range (3300, 4600 *rad/seg*). This Figure shows the absorption of vibration only for the mode most affected by the dynamic loading (*n*=3). However, the interest is the full dynamic response (considering a high number of terms in Expression 9). This is shown in Figure 3. As observed, the original resonance does not appear. However, vibration attenuation is given in a lower range (3400, 4250 *rad/seg*.). This is due to the fact that external frequency ω could be close to another natural frequency of the original system.

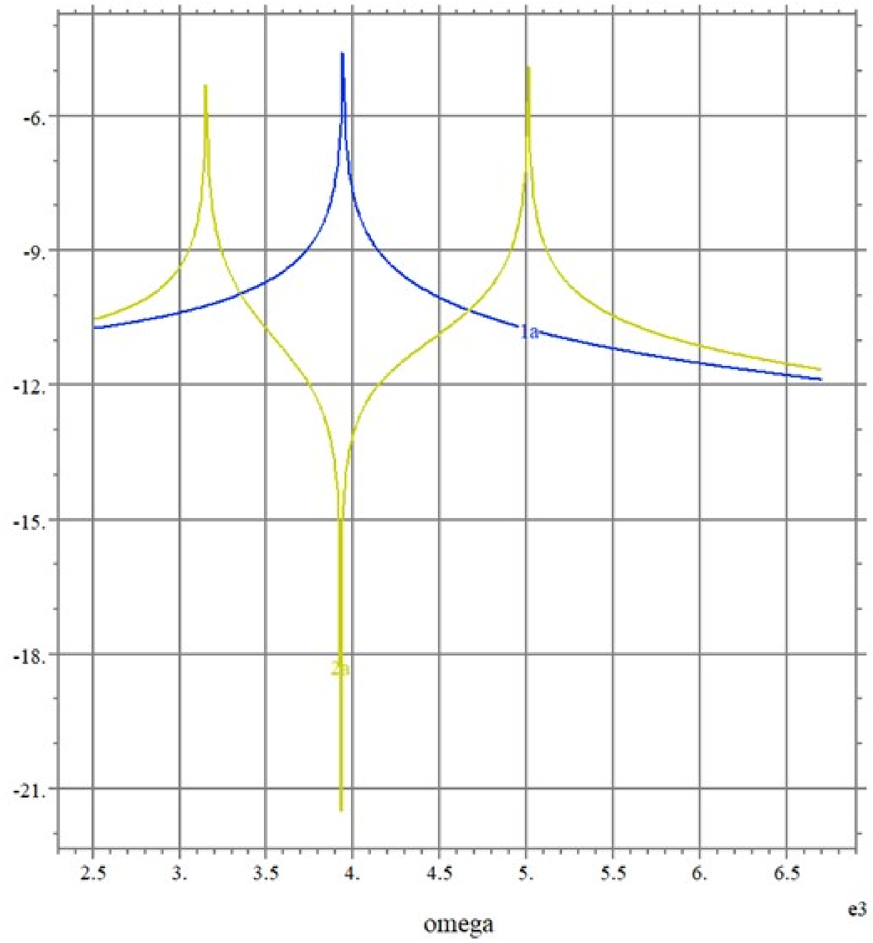

Figure 2: Dynamic response of the beam at x=L/2 for the W_s component n=3: $abs(\alpha sin(3\pi/2))$ (blue line: without resonators, yellow line with resonators, α=0.5)

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3.2 Natural frequencies criterion for locating the bandgap

Another approach to analyse the frequency range of vibration absorption is by studying the natural frequencies of the metamaterial beam with respect to the original beam (without absorbers). In fact, once obtained the natural frequencies ω_{nL} and ω_{nU} limiting the local resonant frequency, it is possible to recognize the vibration attenuation range between these two frequencies because of the non-existence of resonance. However, as explained in connection with Figure 3, one should verify that the external frequency ω does not coincide with any natural frequency associated with another value of *n*. Figure 4 presents the variation of natural frequencies, ω_{nL} and ω_{nU}, as a function of *n* (that is related with the modal wavelength). The painted band shown in Figure 4 indicates the values of the external frequency ω that cannot produce resonance. This is the vibration attenuation frequency range or bandgap. As shown in the Figure, this bandgap corresponds to the range (3550, 4250 *rad/seg*). The first of these values coincides with the local resonance absorber frequency ω_M. This is a general result (as will be explained in the following section) and this is the reason for which, from the point of view of design, ω_M should be slightly lower than ω_T. In fact, bandgap is formed to the high frequency side of the local resonance frequency of the absorber (ω_M).

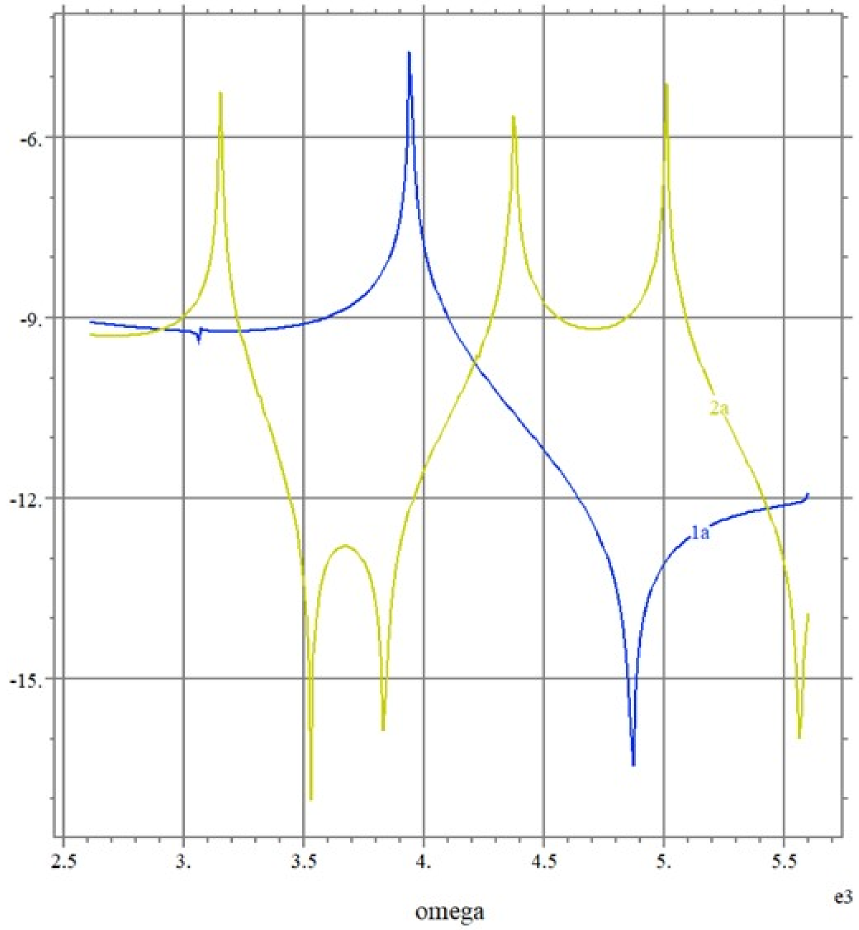

Figure 3: Dynamic response of the beam at x=L/2: $abs(W_s(L/2))$ *(blue line: without resonators, yellow line with resonators,* α*=0.5)*

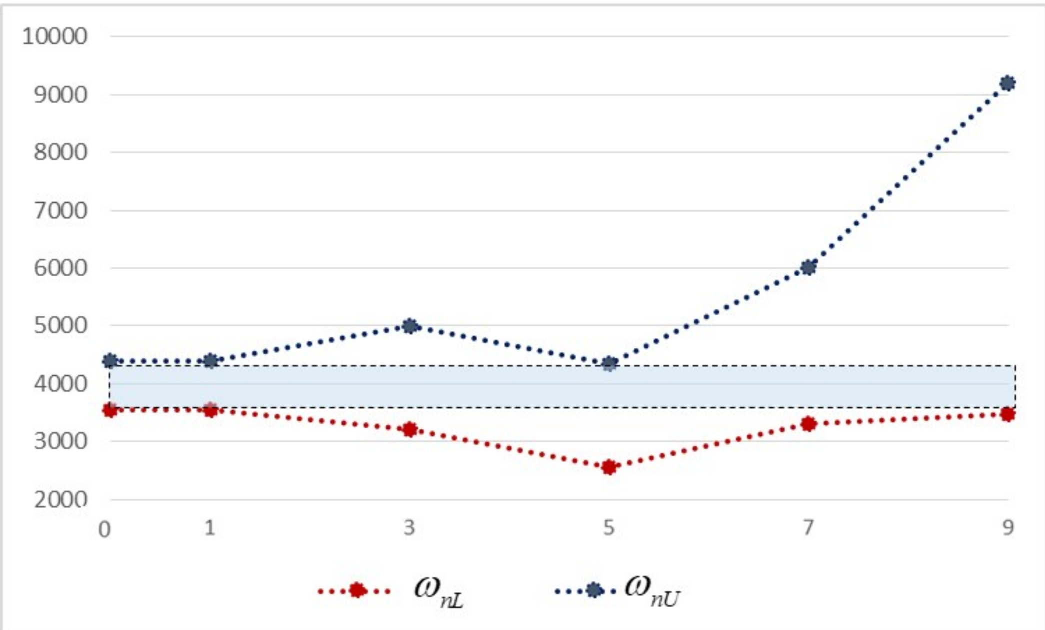

Figure 4: Natural frequencies (ω_{nL} *and* ω_{nU}*) limiting the local resonator frequency (*ω_M*) as a function of the shape modal number n.*

3.3 General approximate procedure for locating the bandgap

It is useful to observe in Figure 4, after interpreting the value *n* as a continuous variable, that an approximation for the frequencies limiting the bandgap may be obtained adopting the values corresponding to *n*=0. These values are 3550 and 4400 *rad/seg*. The first coincides with the local absorber frequency and the second is slightly upper than the bandgap limiting frequency obtained with the above forced vibration analysis. However, this approach for obtaining the bandgap limiting frequencies presents some advantages. To understand such advantages, it is convenient to rewrite Equations 11 (with q_n =0 and m_n =0) in the following matrix form:

$$\left(\mathbf{K}_n - \omega_n^2 \mathbf{M}\right)\mathbf{X} = 0 \tag{17}$$

where:

$$\mathbf{K}_n = \begin{bmatrix} \dfrac{EI\pi^4}{\lambda_n^4} & 0 \\ 0 & \dfrac{EC_w\pi^4}{\lambda_n^4} + \dfrac{GJ\pi^2}{\lambda_n^2} \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} \overline{\rho A} & -\overline{\rho A y_s} \\ -\overline{\rho A y_s} & \overline{\rho I_s} \end{bmatrix}, \quad \mathbf{X}_n = \begin{Bmatrix} \alpha_n \\ \beta_n \end{Bmatrix} \tag{18}$$

It is important to note that when *n*=0, stiffness matrix $\mathbf{K}_n$ becomes a null matrix and then Equation 17 is equivalent to the equation:

$$\left|\mathbf{M}(\omega)\right| = 0 \tag{19}$$

In fact, the roots of this equation are the frequencies limiting approximately the bandgap and correspond to the values shown in Figure 4 for *n*=0. It is important to realize that Equation 19 only depends on the inertial properties of the original beam and not on its stiffness properties (nor on the modal number *n*). In this way, Equation 19 is applicable to beams having arbitrary boundary conditions. It is possible to demonstrate analytically that lower root of Equation 19 is ω_M .

Another justification for the use of Equation 19 may be established by observing the general mathematical structure of Equations 17 and 18. In fact, one can note that $\mathbf{K}_n$ is a positive-definite matrix. In this way, if $\mathbf{M}$ is also positive-definite, the System 17 admits positive eigenvalues ω_n^2. However, if $\mathbf{M}$ is not positive-definite, Equation 17 may not admit positive eigenvalues, which means the non-existence of resonance for the values of ω causing this fact in **M.**

Therefore, it is possible to determine an approximation to the bandgap by calculating the eigenvalues γ of matrix **M:**

$$\left(\mathbf{M} - \gamma \mathbf{I}\right)\mathbf{X} = 0 \tag{20}$$

From the above system one obtains:

$$\gamma_{1,2} = \frac{\left(\overline{\rho A} + \overline{\rho I_s}\right) \mp \sqrt{\left(\overline{\rho A} + \overline{\rho I_s}\right)^2 + 4\overline{\rho A}\,\overline{\rho I_s}\,\overline{\rho A y_s}^2}}{-4\overline{\rho A y_s}^2} \tag{21}$$

If $\gamma_1 < 0$ or $\gamma_2 < 0$, $\mathbf{M}$ is not positive-definite and then the frequencies limiting this zone constitute the approximate bandgap. These last ones correspond to γ_1 and $\gamma_2 = 0$, but for these cases Equation 20 is the same as Equation 19. Thus, the procedure corresponding to Equation 19 is again justified. In Figure 5, the values of the indicator I_M is given as a function of ω. Such an indicator is defined to be 1, if the eigenvalues γ_1 and γ_2 are positive, and 0 otherwise. In this way, the range where $I_M = 0$ corresponds approximately to the bandgap. The limiting frequency values coincide with the roots of Equation 19, ω_M being the lowest limit. It is interesting to note that this procedure reproduces the formula of Sugino et al. for non-coupled flexural or torsional vibrations ($y_s = y_M = 0$).

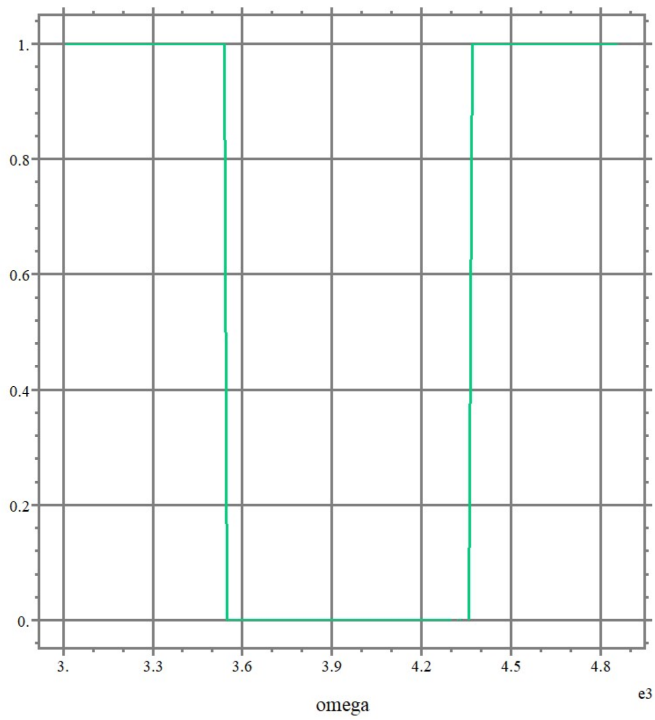

Figure 5: Indicator of the Mass Matrix eigenvalues signs I_M (=1 if γ_1 or γ_2 are positive, and 0 otherwise; Bandgap when I_M =0)

4. CONCLUSIONS

Coupled flexural-torsional dynamics of metamaterial structures consisting of a thin-walled beam with many spring-mass resonators periodically attached is studied. Especial emphasis was done in describing the vibration absorption properties and the bandgap formation. The problem was formulated by means of the Vlasov´s theory. An exact analytical solution is obtained for free and forced vibration for simply supported channel beams under the assumption of continuously distributed absorbers along the length. Three approaches were proposed for analysing the frequency range where attenuation of vibration is produced. The first one is based on the analysis of the dynamic response of the structure under certain dynamic loading. The second one is based on the calculation of the natural frequencies limiting the local resonant absorber frequency as a function of the wavelength (or the modal shape number n). Both methodologies are appropriate to estimate accurately the bandgap location and width. A third approximate methodology was proposed based on the analysis of the inertial properties of the original beam. This last approximate procedure is very simple and can be used for thin-walled beams with arbitrary boundary conditions. Cross sectional shapes with only one axis of symmetry were analysed. However, the methodology may be extended in a straightforward manner for considering completely asymmetric cross-sectional shapes.

5. ACKNOWLEDGEMENTS

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APPENDIX

Numerical Data:

$E = 4.5E10\,Pa,\ G = 1.8E10\,Pa,\ \rho = 2650\,kg/m^3,$

$h = 0.1m, b = 0.04m,\ t = 0.006m,\ L = 2m,\ A = 1.008E-03\,m^2,$

$y_s = -0.020816m,\ \ I_s = 1.97751E-06\,m^4,\ y_M = 0.028m,$

$q = 1000\,N/m,\ m_t = 0,\ \alpha = 0.5.$