MADRID
inter.noise 2019
June 16 - 19
NOISE CONTROL FOR A BETTER ENVIRONMENT

The acoustical coupling of poro-elastic media in a layered structure based on the transfer matrix method

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ABSTRACT

Originally inspired by a classical modeling methodology in electrical engineering, the transfer matrix method (TMM) has proven to be an accurate and efficient way to model layered acoustic media. In the case of fluid, or effective fluid, media, the acoustic TMM elements are conventionally modeled as two-by-two matrices. In contrast, a six-by-six matrix is required to model a poro-elastic layer because of the multiple types of waves that can propagate within it. Introduced here is a modified TMM that draws on various matrix operations to couple the six-by-six poro-elastic layer matrix with the two-by-two matrices of other acoustic elements. The matrix operations mainly include two steps: singular value decomposition and QR decomposition, which allows the order of a poro-elastic layer matrix to be reduced from six-by-six to two-by-two, so that a layered system can be modeled by multiplying together a sequence of two-by-two matrices for all the layered acoustic elements, thus finally creating a "global" two-by-two matrix. In this article, the proposed method was applied to several different layered or multi-panel structures, and the predicted acoustical properties were compared to results obtained by using other existing methods to validate the modified TMM.

Keywords: Acoustical Modeling, Transfer Matrix Method, Poro-Elastic Media, Layered Structure
I-INCE Classification of Subject Number: 30
(see http://i-ince.org/files/data/classification.pdf)

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1. INTRODUCTION

Modeling of multilayered acoustical systems is a classical topic in noise and vibration control engineering. Mason, in his 1927 paper [1], first proposed an acoustical-electrical analogy to use two-by-two transfer matrices to model acoustical lumped elements such as acoustic filters and horns. This idea was later cited in Pierce's book [2] as the concept of "Acoustical Two-Ports", and was further used by different researchers for modeling layered acoustical elements. For example, Lai *et al*. [3] modeled resistive scrims, limp impermeable membranes, stiff panels, air spaces and limp fibrous layers as two-by-two transfer matrices; Thompson [4], Brekhovskikh [5], Folds and Loggins [6] and Scharnhorst [7] modeled elastic solid layers as four-by-four transfer matrices; and Allard *et al*. [8] modeled poro-elastic layers as six-by-six transfer matrices due to the three types Biot waves [9] propagating through the layer. A challenge arises here when coupling multilayered acoustical elements that have transfer matrices of different dimensions. One coupling method was proposed by Brouard *et al*. [10] by assembling boundary conditions between layers into a "global transfer matrix", and by representing the multilayered system in terms of transfer matrix elements, and this method was later summarized in Allard and Atalla's book [11]. The other coupling method that was proposed by Bolton *et al*. [12] consists of constructing the equation system that results from the boundary conditions of a layered structure, and by representing the multilayered system in terms of propagating wave amplitude coefficients. The latter method was also referred to as the "arbitrary coefficient method" (ACM) in recent publications [13–15].

When multiple acoustical lumped elements are combined in a series, a simple way to couple them is to multiply all the transfer matrices together since they share the same two-by-two dimension. Inspired by this idea, presented here is a transfer matrix method (TMM) based on matrix operations including singular value decomposition (SVD) and QR decomposition (QRD), so that the dimension of a six-by-six poro-elastic layer transfer matrix can be reduced to two-by-two, and then be easily coupled with other two-by-two elements. Note that a similar method was introduced by Lauriks *et al*. [16] to explicitly derive a two-by-two matrix from a six-by-six matrix of a double-panel plus poro-elastic layer sandwich structure, but the TMM proposed here is different in that all the matrix operations for dimension reduction were conducted implicitly to simply the process. The matrix dimension reduction process is introduced in Section 2. Further, the proposed TMM was verified by comparison with the ACM, and the results are shown in Section 3.

2. TRANSFER MATRIX DIMENSION REDUCTION FOR A PORO-ELASTIC LAYER

A common configuration of a poro-elastic layer being driven by an oblique incidence plane wave is shown in Figure 1 to help illustrate the general approach of the matrix dimension reduction process.

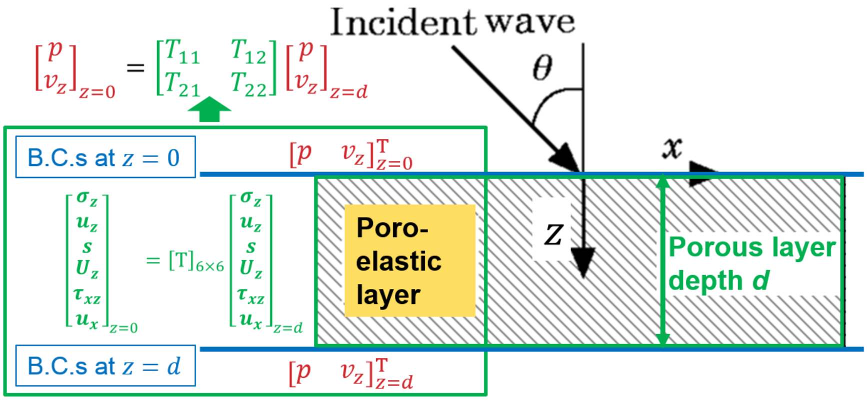

Figure 1: The general approach of the matrix dimension reduction process.

2.1. Introduction of Matrices

First, the vectors of propagating waves (also referred to as "field variables") on both sides of the poro-elastic layer can be connected by a six-by-six transfer matrix, [**T**]: *i.e.*,

$$\begin{bmatrix}\sigma_z & u_z & s & U_z & \tau_{xz} & u_x\end{bmatrix}^{\mathrm{T}}_{z=0} = \begin{bmatrix}\mathbf{T}\end{bmatrix}_{6\times 6}\begin{bmatrix}\sigma_z & u_z & s & U_z & \tau_{xz} & u_x\end{bmatrix}^{\mathrm{T}}_{z=d}, \tag{1}$$

where σ_z is the force per unit material area acting on the solid phase of the poro-elastic medium in the z-direction, s is the force per unit material area acting on the fluid phase of the poro-elastic medium (opposite in sign to a pressure), u_z and U_z are the z-direction displacements for the solid and fluid phases of the poro-elastic medium, respectively, τ_{xz} is the shear force per unit material area acting on the solid phase in the x-z plane, and u_x is the x-direction displacement for the solid phase. The detailed solution for [**T**] was introduced in Refs. [8, 11].

Further, by applying the boundary conditions (B.C.s), the field variables vectors can be connected with the acoustic pressures and particle velocities on both sides of the layer by

$$\begin{bmatrix}\mathbf{M}_1\end{bmatrix}_{4\times 6}\begin{bmatrix}\sigma_z & u_z & s & U_z & \tau_{xz} & u_x\end{bmatrix}^{\mathrm{T}}_{z=0} = \begin{bmatrix}\mathbf{N}_1\end{bmatrix}_{4\times 2}\begin{bmatrix}p & v_z\end{bmatrix}^{\mathrm{T}}_{z=0}, \tag{2}$$

and

$$\begin{bmatrix}\mathbf{M}_2\end{bmatrix}_{4\times 6}\begin{bmatrix}\sigma_z & u_z & s & U_z & \tau_{xz} & u_x\end{bmatrix}^{\mathrm{T}}_{z=d} = \begin{bmatrix}\mathbf{N}_2\end{bmatrix}_{4\times 2}\begin{bmatrix}p & v_z\end{bmatrix}^{\mathrm{T}}_{z=d}. \tag{3}$$

To be more specific, when the layer has open surfaces on both sides, the B.C.s to be satisfied are two normal stress conditions, one normal volume velocity condition and one shear stress condition as described in Ref. [12]: *i.e.*,

$$s = -\phi p, \tag{4a}$$

$$\sigma_z = -(1-\phi)p, \tag{4b}$$

$$i\omega(1-\phi)u_z + i\omega\phi U_z = v_z, \tag{4c}$$

$$\tau_{xz} = 0, \tag{4d}$$

where ϕ is the porosity of the layer, and ω is the angular frequency. Therefore,

$$\left[\mathbf{M}_1\right] = \left[\mathbf{M}_2\right] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & i\omega(1-\phi) & 0 & i\omega\phi & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \tag{5}$$

and

$$\left[\mathbf{N}_1\right] = \left[\mathbf{N}_2\right] = \begin{bmatrix} -(1-\phi) & 0 \\ -\phi & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{6}$$

On the other hand, when the layer is bonded to a stiff panel, the B.C.s to be satisfied include four motion continuity conditions, one shear stress condition and one panel equation of motion as per Ref. [12]: *i.e.*,

$$v_z = i\omega w_t, \tag{7a}$$

$$u_z = w_t, \tag{7b}$$

$$U_z = w_t, \tag{7c}$$

$$u_x = w_p(-/+)ik_x\frac{h_p}{2}w_t, \tag{7d}$$

$$(+/-)\tau_{xz} = \left(D_p k_x^2 - m_s\omega^2\right) w_p, \tag{7e}$$

$$(+/-)p(+/-)s(+/-)\sigma_z - ik_x\frac{h_p}{2}\tau_{xz} = \left(Dk_x^4 - m_s\omega^2\right) w_t, \tag{7f}$$

where $k_x = \omega \sin\theta / c_0$ is the trace wavenumber in terms of incidence angle, θ and the speed of sound in the air, c_0, w_t and w_p are the transverse and longitudinal displacements of the panel, respectively, $m_s = \rho_p h_p$ is the mass per unit area of the panel based on its density, ρ_p, and thickness, h_p, and $D = \left[h_p^3 E_0(1 + i\eta_p)\right] / \left[12(1 - \nu_p^2)\right]$ and $D_p = E_0 h_p$ are the flexural and longitudinal stiffnesses per unit length in the y-direction of the panel, respectively, based on h_p, the Young's modulus, E_0, the loss factor, η_p, and the Poisson's ratio, ν_p. Note that in B.C.s 7d–7f, the first signs are used when the panel is bonded to the incident side of the layer, and the second signs are used for when the panel is bonded to the transmission side of the layer. Therefore, in this case,

$$\left[\mathbf{M}_1\right] = \begin{bmatrix} -1 & Dk_x^4 - m_s\omega^2 & -1 & 0 & ik_x h_p/2 & 0 \\ 0 & -ik_x h_p/2 & 0 & 0 & -1/\left(D_p k_x^2 - m_s\omega^2\right) & 1 \\ 0 & i\omega & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\omega & 0 & 0 \end{bmatrix}, \tag{8a}$$

$$\left[\mathbf{M}_2\right] = \begin{bmatrix} 1 & Dk_x^4 - m_s\omega^2 & 1 & 0 & ik_xh_p/2 & 0 \\ 0 & ik_xh_p/2 & 0 & 0 & 1/\left(D_pk_x^2 - m_s\omega^2\right) & 1 \\ 0 & i\omega & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\omega & 0 & 0 \end{bmatrix}, \tag{8b}$$

and

$$\left[\mathbf{N}_1\right] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \tag{9a}$$

$$\left[\mathbf{N}_2\right] = \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}. \tag{9b}$$

2.2. Singular Value Decomposition (SVD)

The SVD here consists of two steps.

In the first step, the SVD can be applied to [$\mathbf{N}_1$] by

$$\left[\mathbf{N}_1\right]_{4\times2} = \left[\mathbf{U}_1\right]_{4\times4}\left[\mathbf{S}_1\right]_{4\times2}\left[\mathbf{V}_1\right]_{2\times2}^{\mathrm{H}}, \tag{10}$$

and Equation 2 can be rewritten as

$$\left[\mathbf{M}_1\right]_{4\times6}\left[\sigma_z \quad u_z \quad s \quad U_z \quad \tau_{xz} \quad u_x\right]_{z=0}^{\mathrm{T}} = \left[\mathbf{U}_1\right]_{4\times4}\left[\mathbf{S}_1\right]_{4\times2}\left[\mathbf{V}_1\right]_{2\times2}^{\mathrm{H}}\left[p \quad v_z\right]_{z=0}^{\mathrm{T}}, \tag{11}$$

where [$\mathbf{S}_1$] consists of the eigenvalues, λ_1 and λ_2, in the form

$$\left[\mathbf{S}_1\right]_{4\times2} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{12}$$

and an intermediate matrix [$\mathbf{A}$] can be introduced here as

$$\left[\mathbf{A}\right]_{4\times6} = \left[\mathbf{U}_1\right]_{4\times4}^{\mathrm{H}}\left[\mathbf{M}_1\right]_{4\times6} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} & A_{26} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} & A_{36} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} & A_{46} \end{bmatrix}, \tag{13}$$

so that Equation 2 can further be rewritten as

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} & A_{26} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} & A_{36} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} & A_{46} \end{bmatrix} \begin{bmatrix} \sigma_z \\ u_z \\ s \\ U_z \\ \tau_{xz} \\ u_x \end{bmatrix}_{z=0} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\left[\mathbf{V}_1\right]_{2\times2}^{\mathrm{H}}\begin{bmatrix} p \\ v_z \end{bmatrix}_{z=0}. \tag{14}$$

Therefore, a coupling matrix [$\mathbf{C}_1$] can be defined as

$$\left[\mathbf{C}_1\right]_{4\times6} = \begin{bmatrix} [\mathbf{C}_{1a}]_{2\times6} \\ [\mathbf{C}_{1b}]_{2\times6} \end{bmatrix}, \tag{15}$$

where

$$\left[\mathbf{C}_{1a}\right]_{2\times6} = \left[\mathbf{V}_1\right]_{2\times2} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} A_{31} & A_{32} & A_{33} & A_{34} & A_{35} & A_{36} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} & A_{46} \end{bmatrix}, \tag{16}$$

and

$$\left[\mathbf{C}_{1b}\right]_{2\times6} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} & A_{26} \end{bmatrix}, \tag{17}$$

so that

$$\begin{bmatrix} p \\ v_z \\ 0 \\ 0 \end{bmatrix}_{z=0} = \left[\mathbf{C}_1\right]_{4\times6} \begin{bmatrix} \sigma_z \\ u_z \\ s \\ U_z \\ \tau_{xz} \\ u_x \end{bmatrix}_{z=0} . \tag{18}$$

In the second step, the SVD is applied to [$\mathbf{M}_2$] by

$$\left[\mathbf{M}_2\right]_{4\times6} = \left[\mathbf{U}_2\right]_{4\times4} \left[\mathbf{S}_2\right]_{4\times6} \left[\mathbf{V}_2\right]_{6\times6}^{\mathrm{H}}, \tag{19}$$

and Equation 3 can be rewritten as

$$\left[\mathbf{U}_2\right]_{4\times4} \left[\mathbf{S}_2\right]_{4\times6} \left[\mathbf{V}_2\right]_{6\times6}^{\mathrm{H}} \begin{bmatrix} \sigma_z & u_z & s & U_z & \tau_{xz} & u_x \end{bmatrix}_{z=d}^{\mathrm{T}} = \left[\mathbf{N}_2\right]_{4\times2} \begin{bmatrix} p & v_z \end{bmatrix}_{z=d}^{\mathrm{T}}, \tag{20}$$

where [$\mathbf{S}_2$] can similarly be expressed in terms of eigenvalues as

$$\left[\mathbf{S}_2\right]_{4\times6} = \begin{bmatrix} \mu_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_4 & 0 & 0 \end{bmatrix}, \tag{21}$$

and an intermediate matrix [$\mathbf{B}$] can be introduced here as

$$\left[\mathbf{B}\right]_{4\times2} = \left[\mathbf{U}_2\right]_{4\times4}^{\mathrm{H}} \left[\mathbf{N}_2\right]_{4\times2} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \\ B_{41} & B_{42} \end{bmatrix}, \tag{22}$$

so Equation 3 can be rewritten as

$$\begin{bmatrix} \mu_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_4 & 0 & 0 \end{bmatrix} \left[\mathbf{V}_2\right]_{6\times6}^{\mathrm{H}} \begin{bmatrix} \sigma_z \\ u_z \\ s \\ U_z \\ \tau_{xz} \\ u_x \end{bmatrix}_{z=d} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \\ B_{41} & B_{42} \end{bmatrix} \begin{bmatrix} p \\ v_z \end{bmatrix}_{z=d} . \tag{23}$$

In contrast to splitting the matrices as in the first step, here the matrices are given a supplement as

$$\begin{bmatrix} \mu_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \left[\mathbf{V}_2\right]^{\mathrm{H}}_{6\times 6} \begin{bmatrix} \sigma_z \\ u_z \\ s \\ U_z \\ \tau_{xz} \\ u_x \end{bmatrix}_{z=d} = \begin{bmatrix} B_{11} & B_{12} & 0 & 0 \\ B_{21} & B_{22} & 0 & 0 \\ B_{31} & B_{32} & 0 & 0 \\ B_{41} & B_{42} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ v_z \\ X_1 \\ X_2 \end{bmatrix}_{z=d}, \tag{24}$$

where

$$X_1 = \mathbf{V}_2(5,1)\sigma_z + \mathbf{V}_2(5,2)u_z + \mathbf{V}_2(5,3)s + \mathbf{V}_2(5,4)U_z + \mathbf{V}_2(5,5)\tau_{xz} + \mathbf{V}_2(5,6)u_x, \tag{25}$$

and

$$X_2 = \mathbf{V}_2(6,1)\sigma_z + \mathbf{V}_2(6,2)u_z + \mathbf{V}_2(6,3)s + \mathbf{V}_2(6,4)U_z + \mathbf{V}_2(6,5)\tau_{xz} + \mathbf{V}_2(6,6)u_x. \tag{26}$$

Therefore, a second coupling matrix $[\mathbf{C}_2]$ can be defined as

$$\left[\mathbf{C}_2\right]_{6\times 4} = \left[\mathbf{V}_2\right]^{\mathrm{H}}_{6\times 6} \begin{bmatrix} 1/\mu_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/\mu_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/\mu_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/\mu_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} & 0 & 0 \\ B_{21} & B_{22} & 0 & 0 \\ B_{31} & B_{32} & 0 & 0 \\ B_{41} & B_{42} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{27}$$

so that

$$\begin{bmatrix} \sigma_z \\ u_z \\ s \\ U_z \\ \tau_{xz} \\ u_x \end{bmatrix}_{z=d} = \left[\mathbf{C}_2\right]_{6\times 4} \begin{bmatrix} p \\ v_z \\ X_1 \\ X_2 \end{bmatrix}_{z=d} \tag{28}$$

Finally, based on Equation 1, Equation 18 and Equation 28, a relation can be found between the acoustic properties, p and v_z, on both sides of the layer, and it is expressed as

$$\begin{bmatrix} p \\ v_z \\ 0 \\ 0 \end{bmatrix}_{z=0} = \left[\mathbf{C}_1\right]_{4\times 6} \left[\mathbf{T}\right]_{6\times 6} \left[\mathbf{C}_2\right]_{6\times 4} \begin{bmatrix} p \\ v_z \\ X_1 \\ X_2 \end{bmatrix}_{z=d} = \left[\mathbf{T}_{\mathrm{SVD}}\right]_{4\times 4} \begin{bmatrix} p \\ v_z \\ X_1 \\ X_2 \end{bmatrix}_{z=d} \tag{29}$$

2.3. Q-R Decomposition (QRD)

The SVD helped to reduce the six-by-six transfer matrix to a dimension of four-by-four; then the QRD can be applied to $[\mathbf{T}_{\mathrm{SVD}}]$ as

$$\left[\mathbf{T}_{\mathrm{SVD}}\right]_{4\times 4} = \left[\mathbf{Q}\right]_{4\times 4} \left[\mathbf{R}\right]_{4\times 4}, \tag{30}$$

where [**Q**] is orthogonal, and [**R**] is upper-triangle, therefore, Equation 29 can be rewritten as

$$\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \\ D_{31} & D_{32} \\ D_{41} & D_{42} \end{bmatrix} \begin{bmatrix} p \\ v_z \end{bmatrix}_{z=0} = \begin{bmatrix} R_{11} & R_{12} & R_{13} & R_{14} \\ 0 & R_{22} & R_{23} & R_{24} \\ 0 & 0 & R_{33} & R_{34} \\ 0 & 0 & 0 & R_{44} \end{bmatrix} \begin{bmatrix} p \\ v_z \\ X_1 \\ X_2 \end{bmatrix}_{z=d} . \tag{31}$$

where

$$\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \\ D_{31} & D_{32} \\ D_{41} & D_{42} \end{bmatrix} = [\mathbf{Q}]^{\mathrm{H}}_{4\times 4} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} . \tag{32}$$

Further X_1 and X_2 can be solved for in terms of elements in [**D**] and [**R**], and Equation 31 can be rewritten as

$$[\mathbf{E}]_{2\times 2} \begin{bmatrix} p \\ v_z \end{bmatrix}_{z=0} = \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix} \begin{bmatrix} p \\ v_z \end{bmatrix}_{z=d} , \tag{33}$$

where

$$[\mathbf{E}]_{2\times 2} = \left(\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} - \frac{1}{R_{33}} \begin{bmatrix} R_{13} \\ R_{23} \end{bmatrix} \left(\begin{bmatrix} D_{31} \\ D_{32} \end{bmatrix}^{\mathrm{T}} - \frac{R_{34}}{R_{44}} \begin{bmatrix} D_{41} \\ D_{42} \end{bmatrix}^{\mathrm{T}} \right) - \frac{1}{R_{44}} \begin{bmatrix} R_{14} \\ R_{24} \end{bmatrix} \begin{bmatrix} D_{41} \\ D_{42} \end{bmatrix}^{\mathrm{T}} \right) . \tag{34}$$

Finally, the acoustical properties on both sides of the layered structure can be connected by a dimension-reduced two-by-two transfer matrix as

$$\begin{bmatrix} p \\ v_z \end{bmatrix}_{z=0} = [\mathbf{T}_{\mathrm{SVDQR}}]_{2\times 2} \begin{bmatrix} p \\ v_z \end{bmatrix}_{z=d} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} p \\ v_z \end{bmatrix}_{z=d} , \tag{35}$$

where

$$\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} = [\mathbf{E}]^{-1}_{2\times 2} \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix} . \tag{36}$$

Note that the dimension-reduced two-by-two matrix represents the poro-elastic layer when it has open surfaces on both side, while it represents the poro-elastic layer plus the stiff panel(s) when the layer is bonded (glued) to the panel(s).

2.4. Calculation of the Acoustical Properties

Based on the elements in the dimension-reduced transfer matrix, the transmission coefficient, TC, and reflection coefficient, RC, can be calculated as

$$\mathrm{TC} = \frac{2e^{ik_z d}}{T_{11} + T_{12}\cos\theta/(\rho_0 c_0) + T_{21}\rho_0 c_0/\cos\theta + T_{22}}, \tag{37}$$

and

$$\mathrm{RC} = T_{11}\mathrm{TC}e^{-ik_z d} + T_{12}\mathrm{TC}e^{-ik_z d}\cos\theta/(\rho_0 c_0) - 1, \tag{38}$$

where $k_z = \omega\cos\theta/c_0$ and ρ_0 is the air density. Further, the absorption coefficient, AC, and transmission loss, TL, can be calculated by

$$\mathrm{AC} = 1 - |\mathrm{RC}|^2, \tag{39}$$

and

$$\mathrm{TL} = 10\log_{10}\left(1/|\mathrm{TC}|^2\right). \tag{40}$$

3. VERIFICATION OF THE TMM MODEL BY COMPARISON WITH THE ACM MODEL

By giving the bulk properties of the poro-elastic medium (*i.e.*, airflow resistivity, σ, porosity, ϕ, tortuosity, α_∞, bulk density, ρ_b, layer thickness, d, Young's modulus, E_1, mechanical loss factor, η_m, and Poisson's ratio, ν, as listed in Table 1), the panel properties (*i.e.*, h_p, E_0, η_p, ν_p and ρ_p as listed in Table 2), and the ambient properties including the dynamic viscosity of air, η, Prandtl number, B^2, specific heat ratio, γ, air loss factor, η_a, plus ρ_0 and c_0, as listed in Table 3, the AC and TL were calculated for different configurations by using both the classical ACM model and the TMM model proposed here, and the results are shown in Figures 2–5. The comparisons for all the layered structures show excellent agreement, which proves the accuracy of the newly-developed TMM model.

Table 1: Parameters for poro-elastic layer.

σ	ϕ	α_∞	ρ_b	d	E_1	η_m	ν
40000 Rayls/m	0.9871	1.2	13.3 kg/m^3	3 cm	10^6 Pa	0.005	0.3

Table 2: Parameters for aluminum panel.

h_p	E_0	η_p	ν_p	ρ_p
3 mm	7×10^{10} Pa	0.003	0.33	2700 kg/m^3

Table 3: Parameters for the air ambient environment when the temperature is 25 °C.

η	B^2	γ	η_a	ρ_0	c_0
1.846×10^{-5} kg/(s·m)	0.71	1.402	0.0005	1.21 kg/m^3	343 m/s

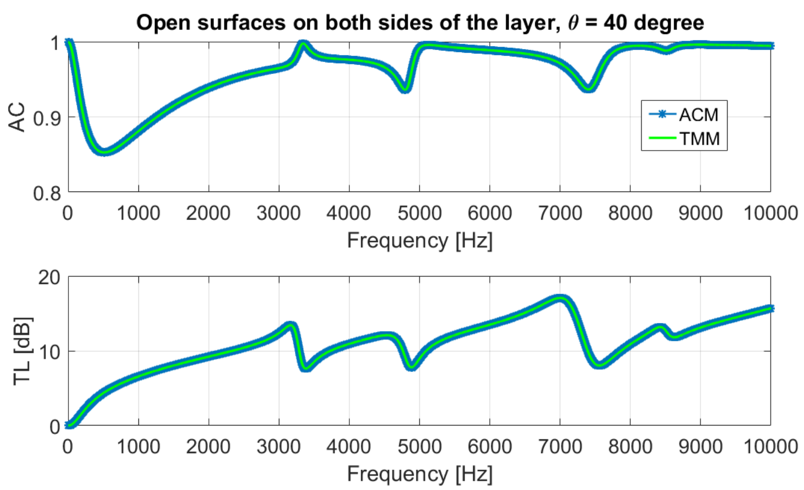

Figure 2: Open surfaces on both sides of the poro-elastic layer.

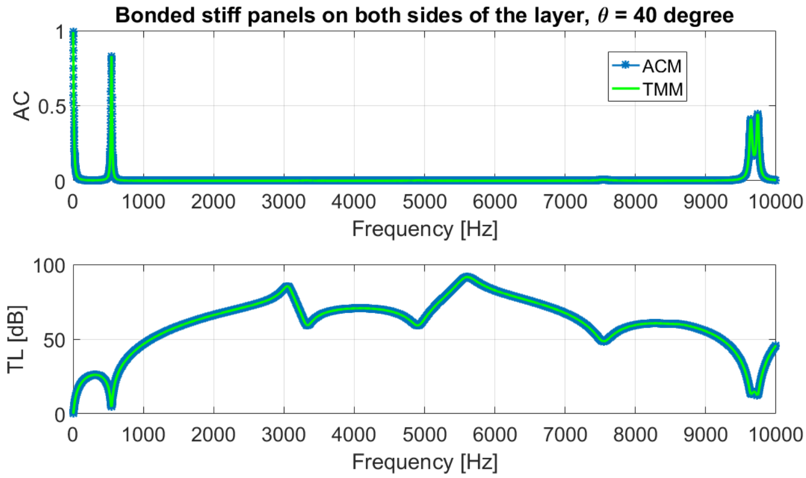

Figure 3: Bonded stiff panels on both sides of the poro-elastic layer.

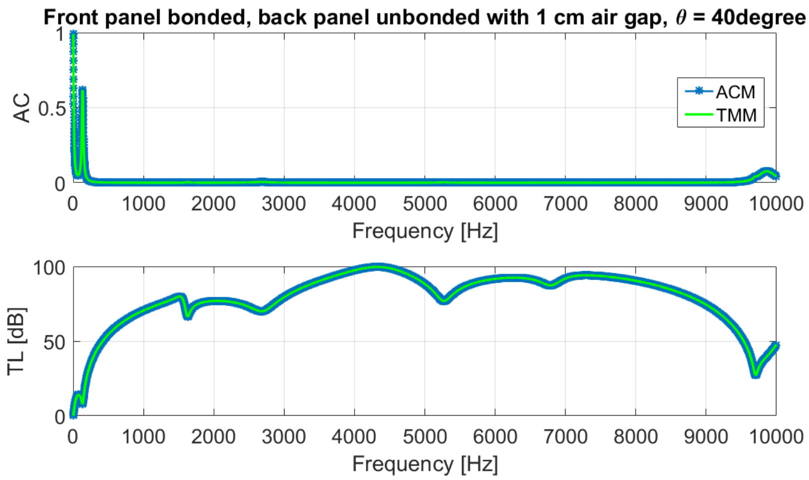

Figure 4: Bonded stiff panel at front surface, and unbonded stiff panel with 1 cm air gap from the back surface of the poro-elastic layer.

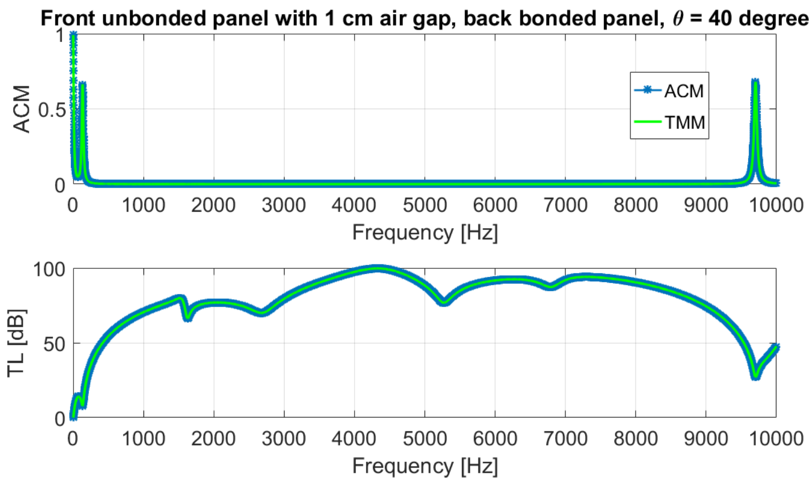

Figure 5: Unbonded stiff panel with 1 cm air gap from front surface, and bonded stiff panel at back surface of the poro-elastic layer.

4. CONCLUSIONS AND FUTURE WORK

A newly-developed transfer matrix method (TMM) for modeling multilayered acoustical systems including poro-elastic layers was proposed in this article. Based on matrix operations including singular value decomposition and Q-R decomposition, the six-by-six transfer matrix of the poro-elastic layer (plus bonded panel(s) if applicable) can be reduced to a dimension of two-by-two, which makes it easier to couple layered acoustic elements that have different matrix dimensions, and improves the modeling efficiency while taking a similar time per calculation. The accuracy of the TMM model was verified by comparison with the classical ACM model [12]. The stabilization of the TMM as suggested in Ref. [17] is going to be completed in the future, which will further extend the capacity of this model especially for extreme cases such as when the layer depth is large. Also, the TMM model will be developed for coupling elastic solid layers, micro-perforated panels and transversely isotropic poro-elastic layers.

5. ACKNOWLEDGMENTS

This research project was financially supported by the 3M Company.

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