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A non-uniform cylindrical shell model for tire-pavement interaction noise simulations

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ABSTRACT

Tire-Pavement Interaction Noise (TPIN) is one of the main pollutants in highly populated areas, especially near highways and main roads. This phenomenon is caused by multiple mechanisms involving the excitation of the tire structure, with a dominant frequency content within 500-1500 Hz. In this work, the theoretical development of a new cylindrical shell model for simulating the vibratory response of a rolling tire is presented. Starting with the classic Donnell-Mushtari-Vlaslov (DMV) theory for orthotropic shells, the equations of motion are simplified. A new single equation defining the radial dynamic behavior of the shell is derived. Orthotropic structural properties are assumed in an effort to accurately replicate those of an actual tire. Non-uniform structural properties along its transversal direction are used to account for differences between its sidewalls and the belt. The effects of rotation and inflation pressure have also been included in the formulation. Finally, the process to compute the response of the tire is presented. This is defined by modes along the transversal direction and waves propagation along its circumferential direction.

Keywords: Tire-Pavement Interaction Noise, Tire Vibrations, Cylindrical Shell Model
I-INCE Classification of Subject Number: 10

1. INTRODUCTION

Tire-Pavement interaction noise (TPIN) is one of the predominant noise sources that contribute to urban pollution. Multiple mitigation strategies are typically implemented to address this problem. For instance, acoustic barriers are built surrounding highways. On the other hand, since 2012, the EU imposed mandatory tire noise regulations that tire manufacturers must comply. As a consequence, there is a need for new modeling methods that accurately predict TPIN for new tire designs [1].

The dominant frequency content for TPIN is within 500-1500 Hz [2]. Many potential

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sources producing this noise have been listed in open literature [3]. However, their actual contribution is subject to discussion, without conclusive proof. Nevertheless, two main categories have been identified: i) noise produced by the vibratory response of the tire, and ii) noise due to air pumping in the contact patch area. In this work, the theoretical developments of a new structural tire model for vibration-induced TPIN are presented.

Empirical data shows that sound intensity levels measured around typical commercial rolling tires decay along their circumferential direction [4]. This suggests the existence of waves propagating and decaying along the circumference of the tire's structure. Further evidence supporting this premise was found by Bernhard [5]. Measured responses showed that above 500 Hz, waves propagating along the circumferential direction of the belt decay rapidly and never travel its complete circumference. This implies that there is not a modal behavior along this direction.

In an effort to simulate tire vibrations and its produced noise, multiple models have been developed. For example, Kropp [6] modeled a tire as a simply supported plate. This approach ignored the tire's curvature since it speculated that it does not influence its response above the characteristic ring frequency. O'Boy [7], implemented a multilayered cylindrical solid model in order to optimize Kropp's structural parameters. On the other hand, the work by Kim et al. [15] introduced a cylindrical shell tire model that accounts for curvature effects and rotation. Finally, Nilsson [8] developed a finite-element-based cylindrical tire model. These approaches provide new methods to predict the tire's response. However, all assume a modal solution along the circumferential direction, thus incorrectly modeling the tire's structural behavior above 500 Hz. In order to overcome this disadvantage, Pinnington [9,10] developed a model where propagating waves are assumed along the circumferential direction of the tire. Yet, the formulation also presents some limitations.

A new cylindrical shell tire model is presented in this paper. The equations of motion of the shell are simplified to account only for the noise emitting radial component. Additional modeling extensions are also presented in an effort to improve accuracy. Finally, a new formulation to compute the response of the tire using a circumferential wave propagation approach is shown.

2. CYLINDRICAL SHELL TIRE MODEL

The structural behavior of a tire is modeled with the cylindrical shell shown in Figure 1. It is assumed that it is simply supported at its two transversal boundaries. This simulates real working conditions where the tire is mounted on a wheel. The dynamic response of the shell is defined in terms of three mid-surface displacement components, in accordance with Kirchhoff's hypothesis [12]. Thus, u corresponds to the transversal displacements along the y-axis, v corresponds to the displacements tangential to the shell's curvature defined by the angle θ, and w corresponds to the radial displacements along the z-axis. In addition, the shell's radius is defined by a, its thickness by h, and its transversal length with L_T. Finally, rotation of the tire can be accounted by the spinning velocity Ω [rad/s].

The structural behavior of the cylindrical shell is governed by the following four kinematic assumptions:

i) The ratio of its thickness to mid-surface curvature must be very small.
ii) Displacements are small if they are compared to the shell's thickness.
iii) Plane sections across the shell thickness remain normal to the mid-surface. This implies that both shear strains parallel to the mid-surface, and those along the radial direction are negligible.
iv) Stresses normal to the mid-surface are assumed to be small compared to other stress components.

This set of assumptions is commonly referred to as the first approximation of shell theory defined by Love. Finally, additional simplifications that follow Donnell-Mushtari-Vlaslov (DMV) theory are implemented in this model. In such case, the mid-surface displacements on the shell's tangent plane, and their derivatives have negligible effects in its curvature and twist [13].

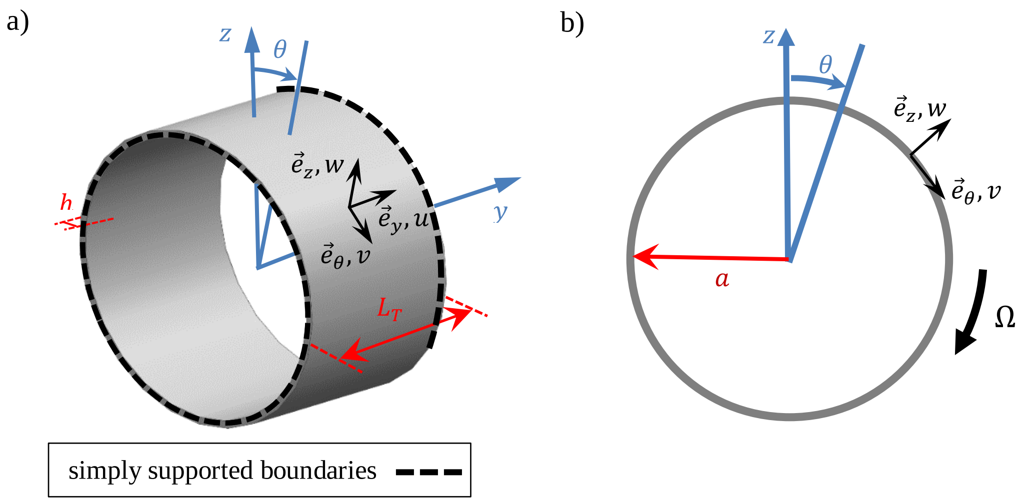

Figure 1. Cylindrical shell model showing coordinate system (blue), geometric parameters (red), and mid-surface displacements (black) in a) three-dimensional view and b) side view.

In accordance with the set of assumptions provided above, a set of fully coupled equations of motion can be obtained for a cylindrical shell with a uniform mass per unit area defied as *m* [12,14]. If the effects of rotation are not yet accounted, these are

$$-\frac{\partial N_y}{\partial y}-\frac{1}{a}\frac{\partial N_{y\theta}}{\partial \theta}+m\frac{\partial^2 u}{\partial t^2}=0 \quad (1)$$

$$-\frac{\partial N_{y\theta}}{\partial y}-\frac{1}{a}\frac{\partial N_{\theta}}{\partial \theta}-\frac{Q_{\theta z}}{a}+m\frac{\partial^2 v}{\partial t^2}=0 \quad (2)$$

$$-\frac{\partial Q_{yz}}{\partial y}-\frac{1}{a}\frac{\partial Q_{\theta z}}{\partial \theta}+\frac{N_{\theta}}{a}+m\frac{\partial^2 w}{\partial t^2}=0 \quad (3)$$

Here, all subscripts correspond to the direction of the resultant forces and moments. N are the normal resultant forces and Q are the shear forces related to the resultant moments M. These are defined as

$$Q_{yz}=\frac{\partial M_y}{\partial y}+\frac{1}{a}\frac{\partial M_{y\theta}}{\partial \theta} \qquad Q_{\theta z}=\frac{\partial M_{y\theta}}{\partial y}+\frac{1}{a}\frac{\partial M_{\theta}}{\partial \theta} \quad (4)$$

A similar approach was implemented by Kim et al. [15] to predict the complete structural response of a rotating tire. However, this approach is not suitable for predictions of vibration-induced noise of a tire. The reason is that the generation of sound by a vibrating surface is dominated by its normal acceleration. This is the only one responsible for the fluid's compression and thus, the radiation of sound [11]. Therefore, a simplified cylindrical shell model is presented in this work, such that curvature of the tire is still accounted but only the radial displacements are kept.

3. STRUCTURAL SIMPLIFICATIONS

This section presents a simplified cylindrical shell model that assumes that its motion is dominated by radial vibrations, responsible for noise emission. Thus, inertia effects on its tangent plane are negligible. Furthermore, it is assumed that the shell's radius is much larger than its thickness. Consequently, shear forces $Q_{\theta z}$ can be neglected as well. In such case, eqn. (1) and eqn. (2) become the following

$$\frac{\partial N_y}{\partial y} + \frac{1}{a}\frac{\partial N_{y\theta}}{\partial \theta} = 0 \qquad \frac{\partial N_{y\theta}}{\partial y} + \frac{1}{a}\frac{\partial N_{\theta}}{\partial \theta} = 0 \tag{5}$$

A shell that follows these simplifications is typically referred to as a shallow shell. However, this denomination is needless. Soedel [16] demonstrated that the simplified approach provides excellent modal agreement if compared to classic Love theory for isotropic shells [12]. The same can be concluded if the shell's transversal stiffness is larger than the circumferential one. This is the case for regular passenger car tires, as shown by the structural properties provided by Pinnington et al. [9]. The only case where this approach presents inaccuracies is if the shell's circumferential stiffness is larger than the transversal one. Such conditions show minor discrepancies for the first transversal breathing mode.

A set of expressions for the resultant forces in eqn. (3) and eqn. (5) are required in order to describe the shell's motion. To obtain them, the following kinematic relationships must be defined

$$\begin{Bmatrix} e_y \\ e_\theta \\ \gamma_{y\theta} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_y \\ \varepsilon_\theta \\ \varepsilon_{y\theta} \end{Bmatrix} + z\begin{Bmatrix} k_y \\ k_\theta \\ \tau \end{Bmatrix} \tag{6}$$

In this case, e_y, e_θ, and $\gamma_{y\theta}$ are the strains at any arbitrary location in the shell. ε_y, ε_θ, and $\varepsilon_{y\theta}$ are the normal and shear strains of the mid-surface. k_y, k_θ, and τ are the mid-surface change in curvature and twist [4]. According to DMV theory, these are given by

$$\begin{aligned} \varepsilon_y &= \frac{\partial u}{\partial y} & k_y &= -\frac{\partial^2 w}{\partial y^2} \\ \varepsilon_\theta &= \frac{1}{a}\left(\frac{\partial v}{\partial \theta} + w\right) & k_\theta &= -\frac{1}{a^2}\frac{\partial^2 w}{\partial \theta^2} \\ \varepsilon_{y\theta} &= \left(\frac{\partial v}{\partial y} + \frac{1}{a}\frac{\partial u}{\partial \theta}\right) & \tau &= -\frac{2}{a}\frac{\partial^2 w}{\partial \theta \partial y} \end{aligned} \tag{7}$$

The stress resultants across the shell can, therefore, be obtained by simply applying Hooke's law written in a tri-dimensional form as

$$\sigma_y = \frac{E}{1-v_i^2}\left(e_y + v_i e_\theta\right) \qquad \sigma_\theta = \frac{E}{1-v_i^2}\left(e_\theta + v_i e_y\right) \qquad \sigma_{y\theta} = G\gamma_{y\theta} \tag{8}$$

where G corresponds to the shear modulus, v_i is the isotropic material's Poison ratio, and E is the isotropic material's modulus of elasticity.

Finally, the force and moment resultants across the shell can be obtained by integrating the stresses eqn. (8) over its thickness, as follows

$$\begin{Bmatrix} N_y \\ N_\theta \\ N_{y\theta} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_y \\ \sigma_\theta \\ \sigma_{y\theta} \end{Bmatrix} dz \qquad \begin{Bmatrix} M_y \\ M_\theta \\ M_{y\theta} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_y \\ \sigma_\theta \\ \sigma_{y\theta} \end{Bmatrix} zdz \tag{9}$$

It is convenient to write the solution to these integrals in matrix form as follows

$$\begin{Bmatrix} N_y \\ N_\theta \\ N_{y\theta} \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} & 0 \\ K_{12} & K_{22} & 0 \\ 0 & 0 & K_{33} \end{bmatrix} \begin{Bmatrix} \varepsilon_y \\ \varepsilon_\theta \\ \varepsilon_{y\theta} \end{Bmatrix} \qquad \begin{Bmatrix} M_y \\ M_\theta \\ M_{y\theta} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{33} \end{bmatrix} \begin{Bmatrix} k_y \\ k_\theta \\ \tau \end{Bmatrix} \tag{10}$$

Here, the resulting forces and moments are now expressed in terms of its strains and stiffnesses. K_{ij} denotes membrane stiffnesses and D_{ij} bending stiffnesses. For the case of an isotropic shell, these stiffnesses are given as

$$\begin{aligned} &K_{11} = K_{22} = K \qquad K_{12} = v_i K \qquad K_{33} = (1-v_i)K/2 \\ &D_{11} = D_{22} = D \qquad \mathrm{D}_{12} = v_i D \qquad \mathrm{D}_{33} = (1-v_i)D/2 \end{aligned} \tag{11}$$

where, D and K are the characteristic isotropic stiffnesses of the cylindrical shell, given as follows

$$K = \frac{Eh^3}{12(1-v_i^2)} \qquad \mathrm{D} = \frac{Eh}{(1-v_i^2)} \tag{12}$$

An alternate approach to define the normal resultant forces is given in the work by Soedel [8]. In this case, these are defined by

$$N_y = \frac{1}{a^2}\frac{\partial^2 \phi}{\partial \theta^2} \qquad N_\theta = \frac{\partial^2 \phi}{\partial y^2} \qquad N_{y\theta} = -\frac{1}{a}\frac{\partial^2 \phi}{\partial \theta \partial y} \tag{13}$$

where, $\phi(y, z, \theta)$ correspond to Airy's stress function [6,8]. In addition, all expressions given in eqn. (13) satisfy eqn. (5). Therefore, they follow the dominant radial vibration assumption for cylindrical shells.

Substituting into eqn. (3), both expressions in eqn. (4), the second expression in eqn. (13), and the resultant moments and strains from eqns. (10) and (7), results in

$$D\nabla^4 w + \frac{1}{a}\frac{\partial^2 \phi}{\partial y^2} + m\frac{\partial^2 w}{\partial t^2} = 0 \tag{14}$$

where $\nabla^4(\cdot) = \frac{1}{a^4}\frac{\partial^4(\cdot)}{\partial \theta^4} + \frac{\partial^4(\cdot)}{\partial y^4} + \frac{2}{a^2}\frac{\partial^4(\cdot)}{\partial y^2 \partial \theta^2}$.

On the other hand, from the six strain-displacement in eqn. (7), the following compatibility equation is found

$$-\frac{1}{a}\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 \varepsilon_\theta}{\partial y^2}-\frac{1}{a}\frac{\partial^2 \varepsilon_{y\theta}}{\partial y \partial \theta}+\frac{1}{a^2}\frac{\partial^2 \varepsilon_y}{\partial y^2}=0 \tag{15}$$

This equation is obtained by performing a set of substitutions, additions, and subtractions using all expressions in eqn. (7) [14, 16]. Furthermore, if the strains in eqn. (15) are expressed in terms of the normal forces defined by eqn. (10), and eqns. (13), then the compatibility equation becomes

$$\frac{Eh}{a}\frac{\partial^2 w}{\partial y^2}-\nabla^4\phi=0 \tag{16}$$

Equations (14) and (16) are the equations of motion for the simplified shell model of a tire. Up to this point, the shell's coupled equations of motion have been reduced from three to two. The two unknowns, in this case, are the shell's normal displacement $w(y,z,\theta)$, and the stress function $\phi(y,z,\theta)$. However, further simplification can be achieved by operating eqn. (14) with the double Laplacian $\nabla^4(\cdot)$, and eqn. (16) with $\partial^2(\cdot)/\partial y^2$. After combining the two, the following equation of motion is obtained

$$\nabla^8 w+\frac{Eh}{a^2}\frac{\partial^4 w}{\partial y^2 \partial \theta^2}-m\omega^2\nabla^4 w=0 \tag{17}$$

Here, it has been assumed that all displacements are harmonic of the form $e^{i\omega t}$. Thus, the last term in the left-hand side of eqn. (17) is multiplied by the squared of frequency ω.

The assumptions made in this section simplified the fully coupled eqns. (1-4) into a single equation of motion, given by eqn. (17). This simplified approach defines the structural motion of a tire solely in terms of the dominant radial displacement. The objective of this process is to avoid unnecessary complications during vibro-acoustic response computations. However, additional improvements are still needed in order to more accurately capture the characteristic structural behavior of a tire. These are addressed in the next section.

4. MODEL EXTENSION FOR TPIN PREDICTIONS

The simplified cylindrical shell tire model is extended in this section. The following set of novel enhancements are implemented:

i) Orthotropic material properties and radial forcing terms in the formulation.
ii) Non-uniform properties along the transversal direction of the tire that account for structural differences between the tire's belt and sidewalls.
iii) Additional membrane tension terms that account for inflation pressure.
iv) Effects of rotation in the tire's structural dynamic behavior.

4.1 Orthotropic Properties and Forcing Terms

The combination of composite materials typically used in a tire results in different structural properties along its circumferential and transversal directions [10]. Thus, a physically accurate model of a tire's structure must account for its orthotropic properties. Accordingly, it is appropriate to change the tri-dimensional Hooke's law given in eqn. (8) to the following

$$\sigma_y=E_y e_y+E_{y\theta}e_\theta \qquad \sigma_\theta=E_\theta e_\theta+E_{y\theta}e_y \qquad \sigma_{y\theta}=G\gamma_{y\theta} \tag{18}$$

where G corresponds to the shear modulus and the orthotropic elasticity moduli are

$$\begin{aligned} E_y &= E_y^{'} \big/ \left(1-\nu_y\nu_\theta\right) \\ E_\theta &= E_\theta^{'} \big/ \left(1-\nu_y\nu_\theta\right) \\ E_{y\theta} &= \nu_\theta E_y^{'} \big/ \left(1-\nu_y\nu_\theta\right) = \nu_y E_\theta^{'} \big/ \left(1-\nu_y\nu_\theta\right) \end{aligned} \tag{19}$$

where the right-hand side terms containing the prime superscript correspond to the effective moduli of the material, while v_y and v_θ are the effective Poisson's ratios [13]. The resultant forces and moments in the shell then become the following

$$\begin{Bmatrix} N_y \\ N_\theta \\ N_{y\theta} \end{Bmatrix} = \begin{bmatrix} E_y h & E_{y\theta} h & 0 \\ E_{y\theta} h & E_\theta h & 0 \\ 0 & 0 & Gt \end{bmatrix} \begin{Bmatrix} \varepsilon_y \\ \varepsilon_\theta \\ \varepsilon_{y\theta} \end{Bmatrix}$$

$$\begin{Bmatrix} M_y \\ M_\theta \\ M_{y\theta} \end{Bmatrix} = \begin{bmatrix} E_y h^3/12 & E_{y\theta} h^3/12 & 0 \\ E_{y\theta} h^3/12 & E_\theta h^3/12 & 0 \\ 0 & 0 & Gh^3/12 \end{bmatrix} \begin{Bmatrix} k_y \\ k_\theta \\ \tau \end{Bmatrix} \tag{20}$$

If the same process to obtain eqn. (14) is followed, this time using the bending stiffnesses in eqn. (20), then the next equation is obtained

$$D_{11}\frac{\partial^4 w}{\partial y^4} + \frac{2\left(D_{12}+2D_{33}\right)}{a^2}\frac{\partial^4 w}{\partial y^2 \partial \theta^2} + \frac{D_{22}}{a^4}\frac{\partial^4 w}{\partial \theta^4} + \frac{1}{a}\frac{\partial^2 \phi}{\partial y^2} + m\frac{\partial^2 w}{\partial t^2} = F_r \tag{21}$$

This is analogous to eqn. (14), but with the orthotropic effects included. In addition, it should be noted that the radial forcing term F_r has also been included.

On the other hand, if the same process to obtain eqn. (16) is followed, but the membrane stiffnesses in eqn. (20) are used, then the following orthotropic compatibility equation is obtained

$$-\frac{1}{a}\frac{\partial^2 w}{\partial y^2} + \alpha_1\frac{\partial^4 \phi}{\partial \theta^4} + \alpha_2\frac{\partial^4 \phi}{\partial y^4} + \alpha_3\frac{\partial^4 \phi}{\partial y^2 \partial \theta^2} = 0 \tag{22}$$

where,

$$\alpha_2 = \frac{K_{11}}{K_{11}K_{22}-K_{12}^2} \qquad \alpha_1 = \frac{K_{22}}{a^4\left(K_{11}K_{22}-K_{12}^2\right)} \qquad \alpha_3 = \frac{1}{a^2}\left(\frac{1}{K_{33}} - \frac{2K_{12}}{K_{11}K_{22}-K_{12}^2}\right) \tag{23}$$

In order to simplify eqn. (21) and eqn. (22) into a single equation of motion for an orthotropic shell, eqn. (21) must now be operated as follows

$$\nabla_D^4\left(\bullet\right) = \alpha_1\frac{\partial^4\left(\bullet\right)}{\partial \theta^4} + \alpha_2\frac{\partial^4\left(\bullet\right)}{\partial y^4} + \alpha_3\frac{\partial^4\left(\bullet\right)}{\partial y^2 \partial \theta^2} \tag{24}$$

On the other hand, eqn. (22) is operated as it was done for eqn. (16). After combining both equations, the resulting single equation of motion for the forced, orthotropic cylindrical shell becomes

$$\chi_1 \frac{\partial^8 w}{\partial y^8} + \chi_2 \frac{\partial^8 w}{\partial y^6 \partial \theta^2} + \chi_3 \frac{\partial^8 w}{\partial y^4 \partial \theta^4} + \chi_4 \frac{\partial^8 w}{\partial y^2 \partial \theta^6} + \chi_5 \frac{\partial^8 w}{\partial \theta^8} + \chi_6 \frac{\partial^4 w}{\partial y^4} \\ -m\omega^2 \left(\alpha_1 \frac{\partial^4 w}{\partial \theta^4} + \alpha_2 \frac{\partial^4 w}{\partial y^4} + \alpha_3 \frac{\partial^4 w}{\partial y^2 \partial \theta^2} \right) = \alpha_1 \frac{\partial^4 F_r}{\partial \theta^4} + \alpha_2 \frac{\partial^4 F_r}{\partial y^4} + \alpha_3 \frac{\partial^4 F_r}{\partial y^2 \partial \theta^2} \tag{25}$$

The multiplying constants in eqn. (25) are given by

$$\chi_1 = \left(D_{11} \alpha_2 \right) \qquad \chi_2 = \left[\alpha_3 D_{11} + \frac{2\alpha_2 \left(D_{12} + 2D_{33} \right)}{a^2} \right] \tag{26}$$

$$\chi_3 = \left[\alpha_1 D_{11} + \frac{\alpha_2 D_{22}}{a^4} + \frac{2\alpha_3 \left(D_{12} + 2D_{33} \right)}{a^2} \right] \qquad \chi_4 = \left[\frac{2\alpha_1 \left(D_{12} + 2D_{33} \right)}{a^2} + \frac{\alpha_3 D_{22}}{a^4} \right]$$

$$\chi_5 = \left(\frac{\alpha_1 D_{22}}{a^4} \right) \qquad \chi_6 = \frac{1}{a^2}$$

4.2 Transversal Non-Uniform Properties

Though the cylindrical shell tire model does not accurately represent a tire's actual transversal geometry, structural differences between the belt and sidewalls need to be accounted for. A smear representation of the tread is used to include its added mass in the tire's belt, as shown in Figure 2. This results in different structural properties between the belt and the sidewalls.

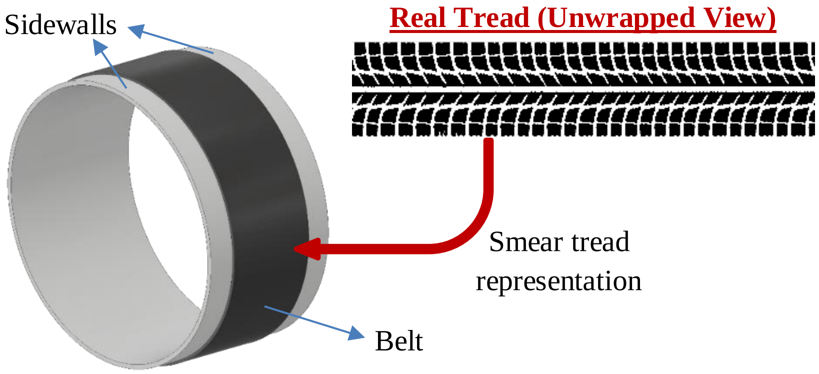

Figure 2. Non-uniform cylindrical shell model with a smear tread representation.

Varying bending stiffness and mass distribution along the tire's transversal direction account for the shell's non-uniformities. These two were selected following the assumption that the radial vibratory motion is dominant. In this case, eqn. (21) becomes the following

$$\frac{\partial^2}{\partial y^2}\left(D_{11}(y) \frac{\partial^2 w}{\partial y^2} \right) + \frac{2\left(D_{12} + 2D_{33} \right)}{a^2} \frac{\partial^4 w}{\partial y^2 \partial \theta^2} + \frac{D_{22}}{a^4} \frac{\partial^4 w}{\partial \theta^4} + \frac{1}{a} \frac{\partial^2 \phi}{\partial y^2} + m(y) \frac{\partial^2 w}{\partial t^2} = F_r \tag{27}$$

As shown in eqn. (27), the bending stiffness D_{11} and the mass m now depend on the transversal direction along the tire i.e., y. Following the same operating process described in the previous sections, eqn. (27) and the compatibility eqn. (22) are used to derive a single equation of motion for an orthotropic non-uniform cylindrical shell, as follows

$$\alpha_1 \frac{\partial^6}{\partial\theta^4\partial y^2}\left(D_{11}(y)\frac{\partial^2 w}{\partial y^2}\right)+\alpha_2\frac{\partial^6}{\partial y^6}\left(D_{11}(y)\frac{\partial^2 w}{\partial y^2}\right)+\alpha_3\frac{\partial^6}{\partial y^4\partial\theta^2}\left(D_{11}(y)\frac{\partial^2 w}{\partial y^2}\right)+$$
$$(\chi_2 - D_{11}(y)\alpha_3)\frac{\partial^8 w}{\partial y^6\partial\theta^2}+(\chi_3 - D_{11}(y)\alpha_1)\frac{\partial^8 w}{\partial y^4\partial\theta^4}+\chi_4\frac{\partial^8 w}{\partial y^2\partial\theta^6}+\chi_5\frac{\partial^8 w}{\partial\theta^8}+$$
$$\chi_6\frac{\partial^4 w}{\partial y^4}-\omega^2\left(\alpha_1 m(y)\frac{\partial^4 w}{\partial\theta^4}+\alpha_2\frac{\partial^4 m(y)w}{\partial y^4}+\alpha_3\frac{\partial^4 m(y)w}{\partial y^2\partial\theta^2}\right)=\alpha_1\frac{\partial^4 F_r}{\partial\theta^4}+\alpha_2\frac{\partial^4 F_r}{\partial y^4}+\alpha_3\frac{\partial^4 F_r}{\partial y^2\partial\theta^2} \quad (28)$$

4.3 Effects of Inflation Pressure

The effects of the inflation pressure are accounted for by the addition of residual membrane tensions along the circumferential and transversal directions of the shell, as shown in the work by Soedel [14, 16]. Their approach was modified to account for these effects in the simplified cylindrical shell. In this case, eqn. (27) is modified to

$$\frac{\partial^2}{\partial y^2}\left(D_{11}(y)\frac{\partial^2 w}{\partial y^2}\right)+\frac{2(D_{12}+2D_{33})}{a^2}\frac{\partial^4 w}{\partial y^2\partial\theta^2}+\ldots$$
$$\frac{D_{22}}{a^4}\frac{\partial^4 w}{\partial\theta^4}+\frac{1}{a}\frac{\partial^2\phi}{\partial y^2}-\frac{N_\theta^r}{a^2}\frac{\partial^2 w}{\partial\theta^2}-N_y^r\frac{\partial^2 w}{\partial y^2}+m(y)\frac{\partial^2 w}{\partial t^2}=F_r \quad (29)$$

where the membrane tension along the transversal direction of the tire have been approximated by $N_\theta^r = pa$, while the circumferential tension has been approximated by $N_y^r = pa/2$. In both cases, p is to the tire inflation pressure [13]. The shell's equation of motion for this case becomes

$$\alpha_1 \frac{\partial^6}{\partial\theta^4\partial y^2}\left(D_{11}(y)\frac{\partial^2 w}{\partial y^2}\right)+\alpha_2\frac{\partial^6}{\partial y^6}\left(D_{11}(y)\frac{\partial^2 w}{\partial y^2}\right)+\alpha_3\frac{\partial^6}{\partial y^4\partial\theta^2}\left(D_{11}(y)\frac{\partial^2 w}{\partial y^2}\right)+$$
$$(\chi_2 - D_{11}(y)\alpha_3)\frac{\partial^8 w}{\partial y^6\partial\theta^2}+(\chi_3 - D_{11}(y)\alpha_1)\frac{\partial^8 w}{\partial y^4\partial\theta^4}+\chi_4\frac{\partial^8 w}{\partial y^2\partial\theta^6}+\chi_5\frac{\partial^8 w}{\partial\theta^8}+$$
$$\chi_6\frac{\partial^4 w}{\partial y^4}-\frac{N_\theta^r}{a^2}\left(\alpha_1\frac{\partial^6 w}{\partial\theta^6}+\alpha_2\frac{\partial^6 w}{\partial y^4\partial\theta^2}+\alpha_3\frac{\partial^6 w}{\partial y^2\partial\theta^4}\right)-N_y^r\left(\alpha_1\frac{\partial^6 w}{\partial y^2\partial\theta^4}+\alpha_2\frac{\partial^6 w}{\partial y^6}+\alpha_3\frac{\partial^6 w}{\partial y^4\partial\theta^2}\right)-$$
$$\omega^2\left(\alpha_1 m(y)\frac{\partial^4 w}{\partial\theta^4}+\alpha_2\frac{\partial^4 m(y)w}{\partial y^4}+\alpha_3\frac{\partial^4 m(y)w}{\partial y^2\partial\theta^2}\right)=\alpha_1\frac{\partial^4 F_r}{\partial\theta^4}+\alpha_2\frac{\partial^4 F_r}{\partial y^4}+\alpha_3\frac{\partial^4 F_r}{\partial y^2\partial\theta^2} \quad (30)$$

4.4 Effects of Rotation

The formulation for the cylindrical shell must also account for effects due to the tire rotation. These are briefly shown in Figure 3.

Figure 3. Wave interaction with the rotation of the tire.

A wave propagation behavior for the mid-frequency range is assumed. In this case, if the tire is excited at the contact region, waves that travel along the circumferential direction of the tire are produced. Those traveling against the rotation of the tire will be slowed down. On the other hand, the velocity of the waves traveling in the same direction as the rotation will be higher. Thus, a difference in wavelength between waves traveling in opposite directions is expected. In this case, the equilibrium equations become

$$\begin{aligned} &-\frac{\partial N_y}{\partial y}-\frac{1}{a}\frac{\partial N_{\theta y}}{\partial \theta}=0 \\ &-\frac{\partial N_{y\theta}}{\partial y}-\frac{1}{a}\frac{\partial N_{\theta}}{\partial \theta}+2m(y)\Omega\frac{\partial w}{\partial t}=0 \\ &-\frac{\partial Q_{yz}}{\partial y}-\frac{1}{a}\frac{\partial Q_{\theta z}}{\partial \theta}+\frac{N_\theta}{a}+m(y)\frac{\partial^2 w}{\partial t^2}-m(y)\Omega^2 w=F_r \end{aligned} \tag{31}$$

Note that these equations include the tire's rotational velocity Ω. Furthermore, since the first two expressions in eqn. (31) need to be satisfied, N_θ previously defined in eqn. (13) changes to the following

$$N_\theta=\frac{\partial^2\phi}{\partial y^2}+2m(y)a\Omega\int\frac{\partial w}{\partial t}d\theta \tag{32}$$

This type of formulation is similar to that proposed by Flügge [17].

If the same process as in the previous sections is followed, but now using eqn. (31) and eqn. (32), the following single equation of motion is obtained

$$\begin{aligned} &\alpha_1\frac{\partial^6}{\partial y^2\partial\theta^4}\left(D_{11}(y)\frac{\partial^2 w}{\partial y^2}\right)+\alpha_2\frac{\partial^6}{\partial y^6}\left(D_{11}(y)\frac{\partial^2 w}{\partial y^2}\right)+\alpha_3\frac{\partial^6}{\partial y^4\partial\theta^2}\left(D_{11}(y)\frac{\partial^2 w}{\partial y^2}\right)+\left(\chi_2-D_{11}(y)\alpha_3\right)\frac{\partial^8 w}{\partial y^6\partial\theta^2} \\ &+\left(\chi_3-D_{11}(y)\alpha_1\right)\frac{\partial^8 w}{\partial y^4\partial\theta^4}+\chi_4\frac{\partial^8 w}{\partial y^2\partial\theta^6}+\chi_5\frac{\partial^8 w}{\partial\theta^8}+\chi_6\frac{\partial^4 w}{\partial y^4}-\omega^2\left(\alpha_1 m(y)\frac{\partial^4 w}{\partial\theta^4}+\alpha_2\frac{\partial^4 m(y)w}{\partial y^4}+\alpha_3\frac{\partial^4 m(y)w}{\partial y^2\partial\theta^2}\right) \\ &-\frac{1}{a^2}\left(\alpha_1 N_\theta^r(y)\frac{\partial^6 w}{\partial\theta^6}+\alpha_2\frac{\partial^6 N_\theta^r(y)w}{\partial y^4\partial\theta^2}+\alpha_3\frac{\partial^6 N_\theta^r(y)w}{\partial y^2\partial\theta^4}\right)-N_y^r\left(\alpha_1\frac{\partial^6 w}{\partial y^2\partial\theta^4}+\alpha_2\frac{\partial^6 w}{\partial y^6}+\alpha_3\frac{\partial^6 w}{\partial y^4\partial\theta^2}\right) \\ &+2i\omega\Omega\left(\alpha_1 m(y)\frac{\partial^3 w}{\partial\theta^3}+\alpha_3\frac{\partial^3 m(y)w}{\partial y^2\partial\theta}+\alpha_4\frac{\partial^3 m(y)w}{\partial y^2\partial\theta}\right)-\Omega^2\left(\alpha_1 m(y)\frac{\partial^4 w}{\partial\theta^4}+\alpha_2\frac{\partial^4 m(y)w}{\partial y^4}+\alpha_3\frac{\partial^4 m(y)w}{\partial y^2\partial\theta^2}\right) \\ &=\left(\alpha_1\frac{\partial^4 F_r}{\partial\theta^4}+\alpha_2\frac{\partial^4 F_r}{\partial y^4}+\alpha_3\frac{\partial^4 F_r}{\partial y^2\partial\theta^2}\right) \end{aligned} \tag{33}$$

This is the equation of motion for an orthotropic, non-uniform, inflated, and rotating shell. Note that, in this case, the left-hand side term that is multiplied by Ω now includes a new constant $\alpha_4=K_{12}/a^2(K_{11}K_{22}-K_{12}^2)$. Furthermore, additional residual stress appears due to a rotation-induced static deflection of the tire [14, 15]. This is added to the circumferential membrane stress coming from the inflation pressure. Therefore, the previously defined $N_\theta^r=pa$ now becomes $N_\theta^r(y)=pa+m(y)a\Omega^2$.

5. WAVE PROPAGATION SOLUTION

Previous work in tire modeling has assumed a full modal solution to compute the response of the tire [6-8]. However, for the frequency range of interest, this approach is no longer physically accurate, as shown in the experimental findings by Bernhard [5]. Therefore, the following wave propagation solution is proposed

$$w(\theta, y, t) = \sum_{n}^{N} q_n \beta_n(y) X_n(\theta) e^{i\omega t} \tag{34}$$

In this case, modes are still assumed along the transversal direction of the tire. The n[th] transversal mode equals the summation of a set of amplitudes and admissible functions, given as $\beta_n(y) = \sum_{m=1}^{M} A_{mn} \Psi(y)$. On the other hand, $X_n(\theta) = e^{-ik_\theta \theta}$ is a wave propagating solution along the circumferential direction of the tire. Finally, q_n correspond to the modal amplitudes. This novel solution approach has already been presented for a flat plate tire model in [18].

If eqn. (34) is substituted into eqn. (33), then pre-multiplied by a vector of the transversal modes, integrated over the transversal direction, and finally, wavenumber transformed, the following system of equations is obtained

$$\begin{bmatrix} [C_1]k_\theta^8 + [C_2]k_\theta^6 + [C_3]k_\theta^4 + [C_4]k_\theta^2 + [\omega_n^2] + \ldots \\ \omega[C_5]k_\theta^3 + \omega[C_6]k_\theta - \omega^2[N_1]k_\theta^4 - \omega^2[N_2]k_\theta^2 - \omega^2[I] \end{bmatrix} \{q_n X_n(k_\theta)\} = \{F_n(k_\theta)\} \tag{35}$$

Here, the matrix $[\omega_n^2]$ is a diagonal matrix containing all the natural frequencies of the system. On the other hand, $[I]$ is a diagonal identity matrix. Finally, matrices$[C_{1\ldots6}]$ and $[N_{1\ldots2}]$ are fully populated matrices. It should also be noted that eqn. (35) is expressed in the wavenumber k_θ.

The next step is to decouple the system of equations. To do so, the off-diagonal terms of all fully populated matrices are ignored. This decoupling assumption is adequate if the matrices' off-diagonal terms are negligible compared its diagonal ones. This is typically the case for an orthotropic tire. The result is the following modal equation

$$q_n X_n(k_\theta) = \frac{F_n(k_\theta)}{C_{1,nn}k_\theta^8 + C_{2,nn}k_\theta^6 + C_{3,nn}k_\theta^4 + C_{4,nn}k_\theta^2 + \omega_n^2 + \omega C_{5,nn}k_\theta^3 + \omega C_{6,nn}k_\theta - \omega^2 N_{1,nn}k_\theta^4 - \omega^2 N_{2,nn}k_\theta^2 - \omega^2} \tag{36}$$

$$\text{for } n = 1, 2, 3 \ldots N \text{ modes and all } k_\theta \text{ from } -\infty \text{ to } +\infty$$

Finally, the inverse wavenumber transform must be computed to define the modal solution in the spatial domain. This results in the following

$$q_n X_n(\theta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{F_n(k_\theta) e^{-ik_\theta \theta}}{C_{1,nn}k_\theta^8 + C_{2,nn}k_\theta^6 + C_{3,nn}k_\theta^4 + C_{4,nn}k_\theta^2 + \omega_n^2 + \omega C_{5,nn}k_\theta^3 + \omega C_{6,nn}k_\theta - \omega^2 N_{1,nn}k_\theta^4 - \omega^2 N_{2,nn}k_\theta^2 - \omega^2} dk_\theta \tag{37}$$

In this case, the modal input force in eqn. (37) is given by

$$F_n(k_\theta) = F_0 e^{ik_\theta \theta_f} \left[\alpha_1 \beta_n(y_f) k_\theta^4 + \alpha_2 \frac{\partial^4 \beta_n(y_f)}{\partial y^4} - \alpha_3 \frac{\partial^2 \beta_n(y_f)}{\partial y^2} k_\theta^2 \right] \tag{38}$$

This expression is found by assuming that the input radial force F_r in eqn. (33) is the following harmonic point force

$$F_r(y, \theta, t) = F_0 \delta(y - y_f) \delta(\theta - \theta_f) e^{i\omega t} \tag{39}$$

where, F_0 is the input force amplitude, and the coordinates (y_f, θ_f) is the location of excitation on the shell's surface.

6. CONCLUSIONS

The characteristic structural properties of a tire define a wave propagation behavior along its circumferential direction. Properly modeling this behavior by accounting for curvature, non-uniform transversal properties, inflation pressure, and rotation is a challenging task. The theoretical work presented in this paper shows a simplified approach to do this while maintaining physical accuracy. In addition, this model is intended to be implemented for TPIN vibration-induced noise. Therefore, the tire's radial displacements have been assumed dominant. This is the first time that such a formulation is proposed. Future efforts include new improvements that account for the proper transversal geometry of a tire. Finite element analysis will be used to do so. Furthermore, noise and vibrations results will be computed in the future to complement this work.

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