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On The Modelling Of Cavities At Low Frequencies With The BEM

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ABSTRACT

Many transducers (e.g. microphones) consist of a moving part facing the exterior on one side and facing a closed cavity on the other. In order to obtain and solve a model of such transducers, a fully coupled model can be used and the performance of this model will rely of the performance of the sub-models being coupled. Previous work has revealed that the performance of an interior lossless BEM formulation concerning a cavity with rigid walls can pose a limit on the accuracy of a fully coupled model of the transducer – in particular when including viscous and thermal losses in the cavity. This paper investigates the accuracy of a lossless acoustic BEM formulation for cavities with rigid walls at low frequencies and addresses possible solutions for restoring accuracy.

Keywords: Numerical Acoustics, Low Frequencies, Accuracy
I-INCE Classification of Subject Number: 71, 76

1. INTRODUCTION

The Boundary Element Method (BEM) has been a popular and valuable tool for calculation models of acoustic transducers like microphones. Initially, a major reason for its use was the easy handling of the infinite domains encountered in e.g. the calculation

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of the free-field response [1]. However, more accurate models are required for very accurate calculations and for the design of transducers, and microphone models are now being developed based on a full coupling of the exterior sound field, the diaphragm and the interior cavity for which viscous and thermal losses are taken into account [2]. The end goal of this research is a unified full model including all relevant effects in the full frequency range. As discussed in Ref. [2] the Finite Element Method (FEM) has also proven capable of handling viscous and thermal losses in acoustic models, but the handling of large and very complex geometries is more of an issue with the FEM than the BEM.

The problem addressed in this paper is the accuracy at very low frequencies. It is well known that dealing with low frequencies in hard-walled cavities is a challenging task since a 'breathing'-type component in the excitation leads to very high sound pressures in hard-walled cavities [3, 4]. Even though this behavior is entirely physically based, it poses challenges for the numerical implementation of problems involving cavities with rigid walls, since the solution becomes very sensitive to any imprecisions in the excitation. Therefore, very high precision of all parts of a coupled formulation of e.g. the microphone problem is required in contrast with the intuitive reasoning, that since the pressure inside the cavity is almost constant with respect to the spatial coordinates, very few degrees of freedom are required to model the sound field in the cavity.

An axisymmetric BEM implementation using the direct method and collocation [5, 6] is used to investigate the 'cavity problem' in setups in which viscous and thermal losses and coupling to the diaphragm and the external sound field are excluded in order to focus the analysis.

2. THE DIRECT BOUNDARY ELEMENT METHOD

The BEM approach to acoustic radiation and scattering problems is based on the Helmholtz Integral Equation that relates the pressure $p(Q)$ and normal velocity $v_n(Q)$ on the surface of a body of any shape with the pressure at any point $p(P)$ and the pressure of an incoming wave $p_{inc}(P)$. The harmonic time dependence $e^{j\omega t}$ is omitted, giving

$$C(P)p(P) = \int_S \left(\frac{\partial G}{\partial n} p(Q) + jk\rho c v_n(Q) G \right) \mathrm{d}S \, + 4\pi p_{inc}(P), \tag{1}$$

where S is the surface of the body, Q a point on that surface and P any exterior or interior point. The normal vector $\mathbf{n}$ is directed into the computational domain. The factor $C(P)$ is the geometrical constant and represents the solid angle at P. The Green's function for 3-D free space is

$$G(R) = \frac{\mathrm{e}^{-jkR}}{R}, \; R = |P - Q|. \tag{2}$$

For fully axisymmetric problems a change to polar coordinates (r, θ) allows for the rewriting of the surface integral to an integral along the generator L of the domain and a rotational integral [6],

$$C(P)p(P) = \int_L \left(p(Q) \int_0^{2\pi} \frac{\partial G}{\partial n} \mathrm{d}\theta + jk\rho c v_n(Q) \int_0^{2\pi} G \mathrm{d}\theta \right) r(Q) \, \mathrm{d}L + 4\pi p_{inc}(P), \tag{3}$$

in which the Jacobian $r(Q)$ is introduced and use of the axisymmetry of $p(Q)$ and $v_n(Q)$ has been made.

Standard collocation results in a matrix equation [5],

$$0 = \mathbf{Ap} + \mathrm{j}k\rho c\mathbf{B}\mathbf{v}_n + 4\pi\mathbf{p}_{inc}, \tag{4}$$

where $\mathbf{p}$ is a vector of pressures at the surface nodes, $\mathbf{v}_n$ is a vector of the normal velocity at surface nodes and $\mathbf{p}_{inc}$ is a vector of incoming pressure at surface nodes.

Setting up a calculation in BEM by placing a monopole of volume velocity Q_m in the geometrical center of an enclosure with rigid walls leads to solving the following equation

$$\mathbf{Ap} = -4\pi\mathbf{p}_{inc}, \tag{5}$$

where $\mathbf{p}_{inc}$ contains the nodal values of

$$p_{inc} = \mathrm{j}\omega\rho Q_m \frac{\mathrm{e}^{\mathrm{j}kR}}{4\pi R}, \tag{6}$$

where R is the distance between the point source and the node. It is evident that the magnitude of the incoming sound field is directly proportional to ω.

In the limit of low frequencies and assuming no losses the sound pressure in the enclosure becomes uniform with a magnitude given by the volume velocity, Q_m, the cyclic frequency ω and the volume of the enclosure V [7],

$$p = \frac{\rho c^2 Q_m}{\mathrm{j}\omega V}. \tag{7}$$

Hence, the sound pressure increases at low frequencies as $1/\omega$.

In order for the solution of Equation (5) to obtain the $1/\omega$ behavior in the limit of low frequencies with a right-hand side increasing as ω, the condition number of $\mathbf{A}$ must show a $1/\omega^2$ behavior at least.

Figure 1 shows the magnitude of the analytical and the BEM calculated sound pressure as well as the condition number as functions of the dimensionless frequency ka for a rigid sphere of radius a (left part of the figure) and for a circular cylinder of radius a and height $2a$ (right part of the figure). The analytical solution is the low frequency approximation of Equation (7) and the excitation is a point source with unit volume velocity placed in the centre of the cavity. The analytical and the BEM solution coincides in the plot and both the $1/\omega$ behavior of the solution and the $1/\omega^2$ behavior of the condition number is confirmed.

3. THE SINGULAR VALUE DECOMPOSITION

The singular value decomposition (SVD) is used to inspect the properties of the $\mathbf{A}$ matrix. The SVD decomposes $\mathbf{A}$ into a product of three matrices:

$$\mathbf{A} = \mathbf{USV} \tag{8}$$

in which U and V are unitary matrices: $\mathbf{UU^H} = \mathbf{I}$ and $\mathbf{VV^H} = \mathbf{I}$ where the superscript H denote conjugate transpose. Matrix $\mathbf{S}$ is a diagonal matrix, with its diagonal values σ_{ii} arranged in descending order. The condition number of $\mathbf{A}$ is the ratio of the largest (first) to the smallest (last) singular value. Figure 2 shows the singular values as a function of dimensionless wavenumber and their index for the sphere (left) and the cylinder (right). It is evident that the increase of the condition number observed in Figure 1 is coursed by

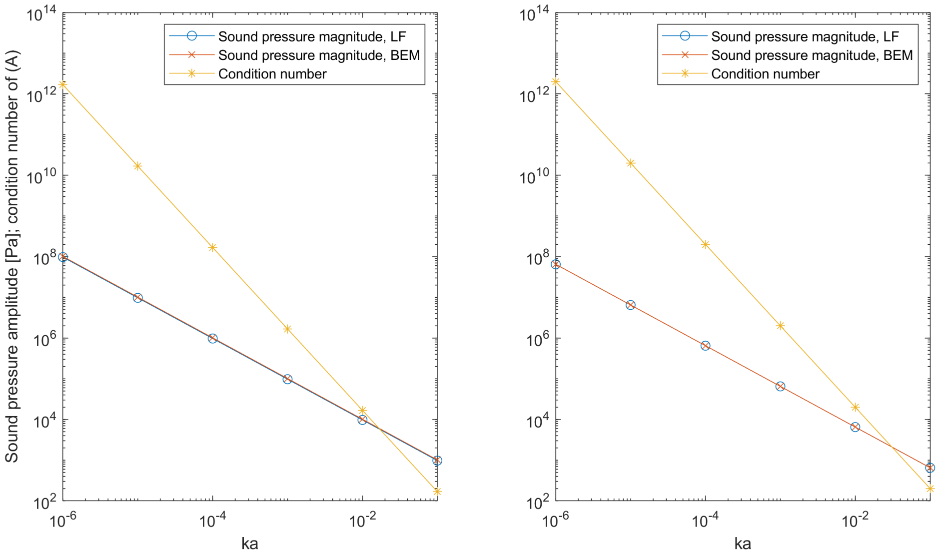

Figure 1: Sound pressure amplitude and condition number as function of dimensionless frequency ka for a cavity excited by a point source of unit volume velocity. Left: spherical cavity of radius a; right: cylindrical cavity of radius a and height 2*a.*

a decrease of the smallest singular value only - all other singular values are practically constant in the range of frequencies observed. Therefore, the increase of the pressure at low frequencies is directly linked to a decrease of the smallest singular value.

In the limit of low frequencies, the oscillating part of the kernel in Equation (3) vanishes and the matrices in Equation (4) become real. Theoretically, the matrix **A** becomes singular as the smallest singular value becomes zero. It follows that the corresponding last column in **V** becomes the eigenvector corresponding to the eigenvalue zero. For a hard-walled cavity, it is well-known that this eigenvector correspond to a constant pressure, and that this eigenvector is orthogonal to all other eigenvectors of the system. Inspection of the last column of matrix **V** shows that this eigenvector does not change rapidly with frequency at low frequencies, which mean that the contribution of the cavity mode to the total solution is largely determined by the last column in **V**.

Since, matrices **U** and **V** are unitary their inverse equal their complex transpose, and once the SVD of **A** has been obtained, the solution of the system of equations $\mathbf{Ax} = \mathbf{b}$ is readily found:

$$\mathbf{x} = \mathbf{V^H S^{-1} U^H b} \tag{9}$$

where $\mathbf{S^{-1}}$ is diagonal matrix containing the reciprocal of the singular values, $1/\sigma_{ii}$. The vector $\mathbf{U^H b}$ is termed the solution coefficients, and due to the properties of $\mathbf{S^{-1}}$, where the last entry is several magnitudes above all other entries, the last entry of $\mathbf{U^H b}$ becomes important for the behavior of the system at low frequencies.

Figure 3 shows the singular value, the solution coefficients and the resulting error as functions of the dimensionless wavenumber for two test cases where analytical solutions are available: a pulsating and an oscillating sphere. Two calculations are shown using the axisymmetric formulation with nine nodes using linear and quadratic elements respectively. It is worth to note, that the **A** matrix is the same independent of the excitation, and that the behavior of the linear and the quadratic formulation is the same, with the exception that the resulting error is more than one magnitude smaller for

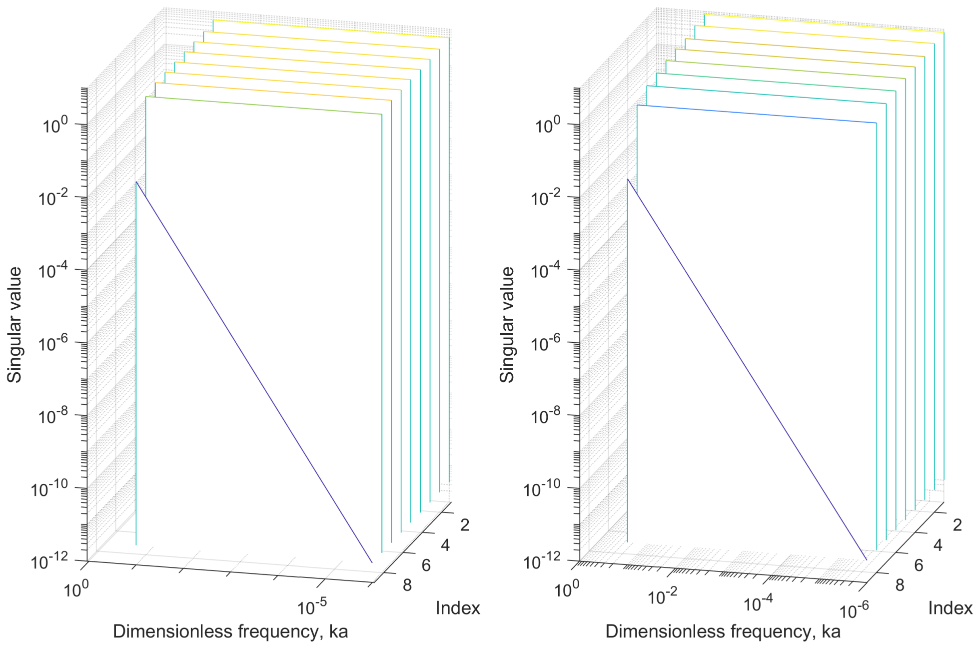

Figure 2: Singular values as function of dimensionless frequency ka for the BEM coefficient matrix A. Left: spherical cavity of radius a; right: cylindrical cavity of radius a and height $2a$.

the quadratic formulation, since the geometry of the sphere is much better represented with quadratic elements. Note, that the error for the pulsating sphere is at a consistent low level for both formulations whereas the error of the oscillating sphere increase to more than 100 percent at low frequencies.

In the limit of low frequencies, the solution coefficient of the pulsating sphere tends to a finite value several orders of magnitude above the last singular value. This results in the solution being by far dominated by the corresponding cavity mode. On the other hand, the oscillating sphere does not in theory excite the cavity mode (since the volume velocity of this vibration should integrate to zero). Indeed the solution coefficient is quite small and decreases with decreasing frequency but not as fast as the last singular value. Hence, for very low frequencies, the solution of the oscillating sphere is 'polluted' with the cavity mode leading to the errors shown. Note, that due to the very high condition number at low frequencies, corresponding to very small values of the last singular value, this effect is important even at very small values of the solution coefficient. Therefore, the calculation is sensitive to even very small errors in the calculation of the matrices - any numerical imprecision can potentially lead to very large errors.

4. HANDLING THE CAVITY MODE WITH THE SVD

The observations that the cavity mode is connected to the last singular value and that it is orthogonal to all other modes in the cavity may lead to a possible remedy of the low frequency difficulty. For the oscillating sphere, the cavity mode should not be present at all. Having the SVD of matrix **A** allows for removing the cavity mode simply by putting the reciprocal of the corresponding singular value to zero: $1/\sigma_{MM} = 0$ before calculating the solution with Equation (9). Figure 4 shows the analytical and numerical (with and

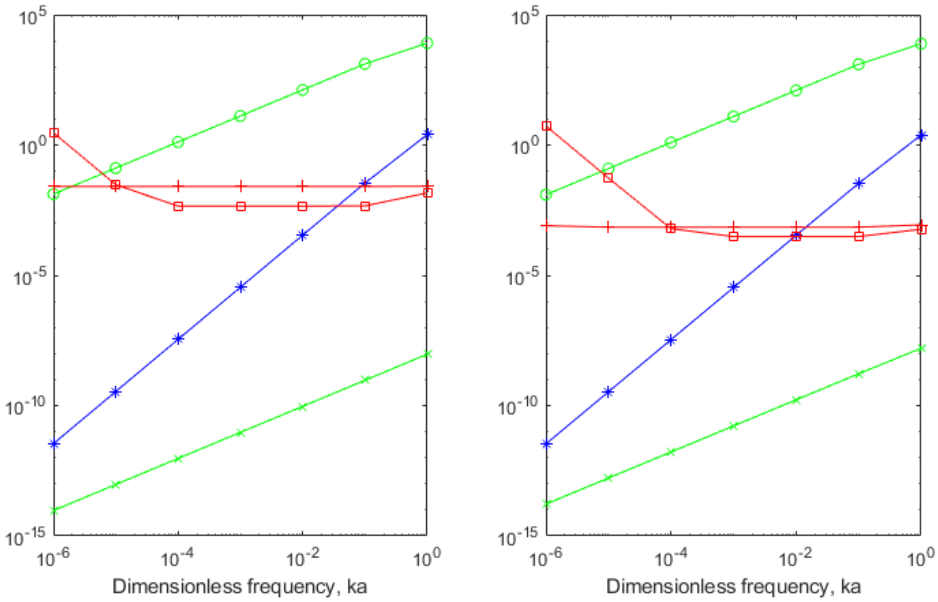

Figure 3: Singular values and solution coefficients of a spherical cavity with pulsating or oscillating walls as function of dimensionless frequency ka. --, Singular values; -o-, solution coefficients for the pulsating sphere; -x-, solution coefficients for the oscillating sphere; -+-, relative error for the pulsating sphere; -□-, relative error for the oscillating sphere. Left: Linear elements, 9 nodes and 8 elements; right: Quadratic elements, 9 nodes and 4 elements*

without the remedy) solution on the interior surface of the sphere at the frequency $ka = 10^{-6}$ as well as the relative error as a function of the frequency. It is evident that the original solution is 'polluted' with an amount of the cavity mode and that subtracting this mode lead to a good solution.

The final test case is a step towards a more realistic senario for calculating the sensitivity of a condenser microphone. A cylindrical cavity with rigid walls of radius $a = 10$ *mm* and a height of 20 μm is considered. Such a geometry is representative for a one inch microphone and besides the small volume it also gives rise to the problem of close surfaces, which has to be carefully dealt with [8]. The cavity is excited by the movement of one of the plane end-faces of the representing the diaphragm. The movement is given by,

$$u_n(r) = u_0\left(\mathrm{J}_0\left(\frac{z_0}{a}r\right) - z_1\right), \tag{10}$$

in which u_0 is an arbitrary amplitude, $z_0 \approx 3.83$ is the first zero of the derivative of the cylindrical Bessel function J_0 and $z_1 \approx -0.40$ is the value of J_0 at z_0. In other words: the prescribed velocity follows the movement of a diaphragm so that the displacement and its derivative with respect to the radial direction is zero at the rim. The test case has been selected so that an analytical solution in terms of two modes can easily be calculated, and the BEM calculations are made using linear elements with mesh sizes from 7 to 259 nodes at a frequency of 54 Hz corresponding to $k = 1\ m^{-1}$.

Having a discretized geometry and a prescribed normal velocity, it is a simple matter to calculate the volume and the volume velocity injected in the cavity. Thereby, the sound pressure due to the cavity mode is easily calculated by Equation (7). Figure 5 shows the error as a function of the mesh fineness (in terms of nodes) for the original BEM

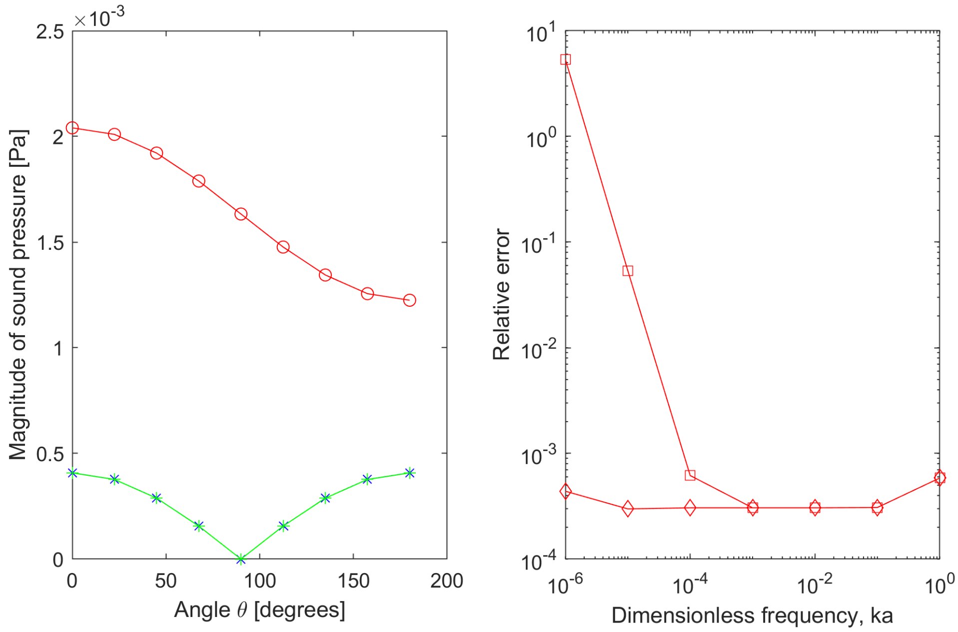

Figure 4: Solution of an oscillating sphere at low frequencies. Left: Magnitude of of the sound pressure as a function of the angle at $ka = 10^{-6}$; --, analytical solution; -o-, original numerical solution; -+-, numerical solution with the cavity mode removed. Right: relative error as a function of dimensionless frequency; -□-, original solution; -◇-, solution with the cavity mode removed. Quadratic elements, 9 nodes and 4 elements*

formulation and for the remedy suggested, in which the cavity mode is removed from the BEM solution as described above, and then added back by the expression in Equation (7). It is evident that the proposed strategy is much more numerically stable and that is restores the convergence of the linear BEM formulation.

5. DISCUSSION AND CONCLUSION

The accuracy of the Boundary Element Method at low frequencies in lossless cavities has been addressed with an axisymmetric BEM implementation using linear and quadratic elements. Using the Singular Value Decomposition on the BEM coefficient matrix, it has been shown that the physical behavior of the sound pressure - i.e. the increase of sound pressure at low frequencies due to the 'cavity mode' - is reflected in a rapid decrease of the last singular value. The corresponding right singular vector is also the eigenvector - i.e. the discretized representation of the cavity mode. Even for problems where there theoretically is no 'breathing' component in the excitation - and therefore no theoretical component of the cavity mode - the strong ill-conditioning of the BEM coefficient matrix courses very small imprecisions to 'excite' the cavity mode, and thereby corrupt the solution.

In the problems studied here, the interior BEM domain is not coupled to other domains or a model of the diaphragm. Therefore, errors are only found at very low frequencies. In coupled problems the level of numerical approximation and imprecision is expected to be larger, and it can be expected that the low frequency problem will be more severe and be found at higher frequencies. On the other hand it is worth to note, that the problem is not a numerical breakdown, in the sense that the behavior observed is physically based.

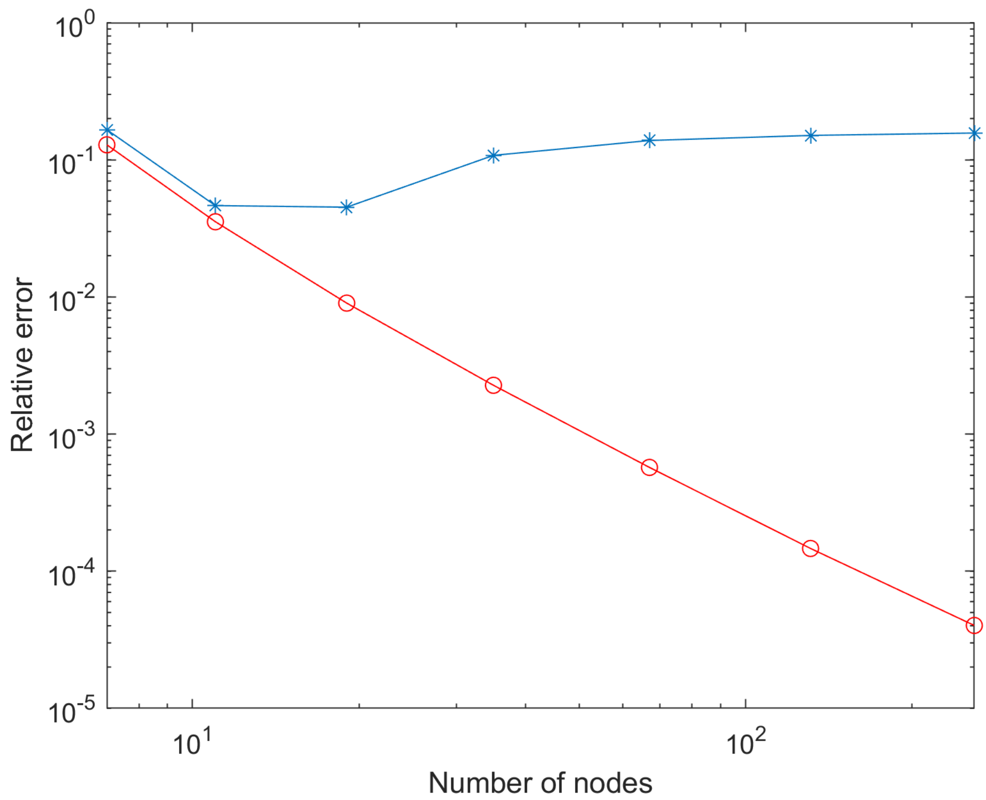

Figure 5: Relative error as function of the number of nodes when solving for the interior sound pressure of a cylindrical cavity of radius $a = 10$ mm and height $h = 20$ µm with one end-face vibrating. --, using the original formulation; -o-, substracting and adding back the cavity mode. The frequency is about 54 Hz corresponding to a wavenumber of* $k = 1\text{m}^{-1}$

Therefore, any numerical model or method that deals with calculation of lossless cavities at low frequencies, can potentially suffer of similar issues.

A possible remedy is to subtract the cavity mode by removing it in the SVD before calculating the matrix inverse, and then to add the component back calculated analytically in the limit of low frequencies. In the problems considered in this paper, this has been successful in a range of frequencies below the first non-trivial mode, but it should be noted that this strategy is not expected to be successful in the full range of frequencies.

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